Geometrical Representation of Election Results

By Sophie Gorman

The Catholic University of America

Goal of voting theory: to identify and select what voters want by means of a social choice function, or voting method

Note that voting theory takes into account a voter's complete preference rankings, C_1 through C_n.

M=total number of voters

Condorcet winner: a candidate who wins every pairwise comparison, or head-to-head race

Condorcet winners don't always exist, and when they do, they don’t always win.

Example:

V_1	V_2	V_3
C_1	C_3	C_2
C_2	C_1	C_3
C_3	C_2	C_1

$C_1 > C_2$

$C_2 > C_3$

$C_1 < C_3$

A loss of transitivity ($C_i > C_i$) is called a Condorcet Cycle

Criteria: standards for a voting method

Examples:

Neutrality

Anonymity

Pareto: If all voters prefer C_i to C_j, then C_j may not win if C_i doesn't as well.

...and many more

Three compelling criteria:

Pareto

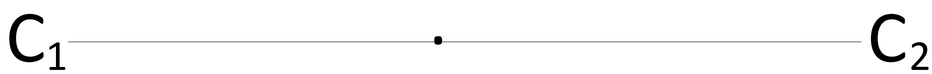
Anonymity

Independence of Irrelevant Alternatives: How C_i and C_j fare compared to each other is irrelevant to how C_k fares

Arrow's Theorem (1951)

A voting system cannot satisfy anonymity, independence, and Pareto at the same time.

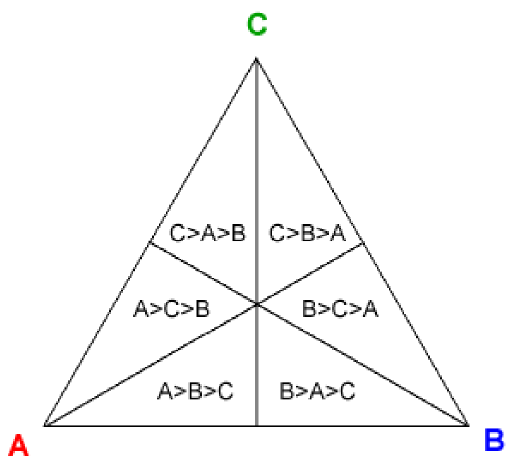
Two candidates:



Region: strict preference region (one possible ranking of candidates)

Populated region: one with at least one voter who chose that ranking

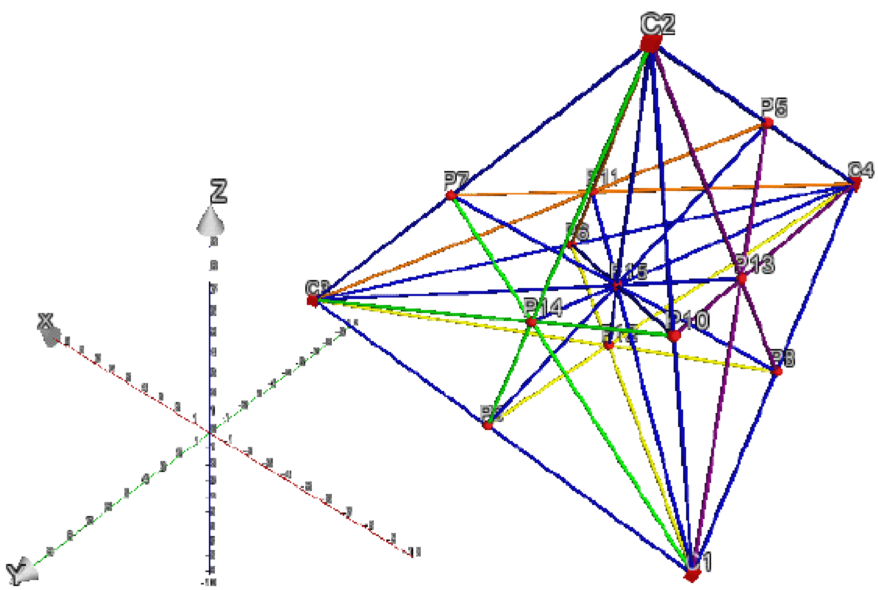
Three candidates:



Barycentric subdivision into six regions

(Image from http://mathdl.maa.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=1195&pf=1)

Four candidates:



Barycentric subdivision into 24 regions

Condorcet domains for three candidates

Proposition: Suppose a profile has ≤ 2 populated regions. Then no loss of transitivity will occur.

Equivalent statement: At least 3 populated regions are necessary to produce a loss of transitivity.

Definition: Condorcet Domain

A Condorcet Domain is one of two subsets:

The union of $C_1 > C_2 > C_3$, $C_2 > C_3 > C_1$, $C_3 > C_1 > C_2$.

The union of $C_1 > C_3 > C_2$, $C_3 > C_2 > C_1$, $C_2 > C_1 > C_3$.

Proposition:

1. If each region of the Condorcet Domain is populated by an equal number of votes, a loss of transitivity will occur.
2. These domains are the only unions of regions which satisfy this property.

Proposition [G]:

Assume all votes are in a Condorcet Domain.

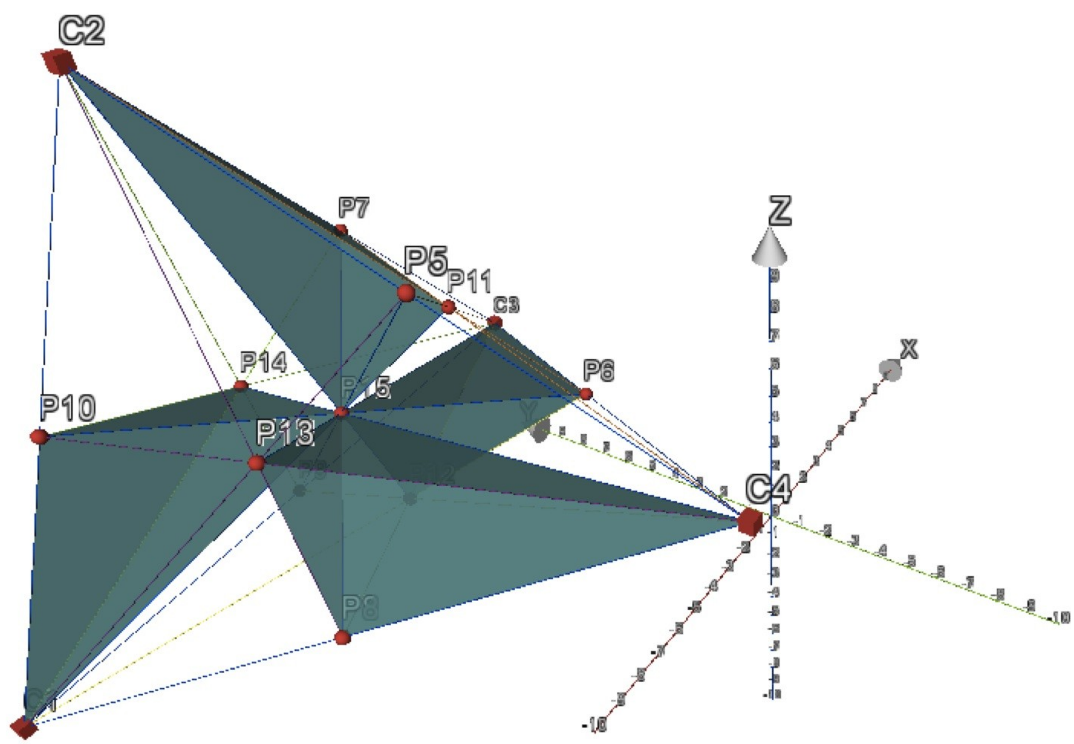
Let N_i=the number of votes in a SPR

$N_i < M/2$ for all N_i if and only if a Condorcet cycle will occur.

Condorcet domains for n candidates

A Condorcet Domain is the union of the SPRs obtained by cyclicly permuting n times a linear ordering of candidates.

For example:



How many Condorcet Domains exist in an election with n candidates?

Proposition [G]:

Total number of SPRs/n

$= n!/n$

$=(n-1)!$

Ideas for further study:

Condorcet cycles with populated regions outside of Condorcet Domains

Condorcet cycles with <n candidates

Deeper insight into voting paradoxes, especially Arrow's Theorem

Sources:

Basic Geometry of Voting and "Condorcet Domains: A Geometric Perspective" by Donald G. Saari