Sums of squares, the octonions, and $(7, 3, 1)$

Ezra Brown
Virginia Tech

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What To Expect

- Sums of squares
- Normed algebras
- $(7, 3, 1)$
- 1-, 2- and 4-square identities and their algebras
- 8 squares
- The octonions
- The connection with $(7, 3, 1)$

A question about sums of squares

Sums of squares

- For which n can the product of two sums of n squares always be written as a sum of n squares?
- Answer (A. Hurwitz, 1898): For $n = 1, 2, 4$ and 8 — and for no other positive integers.
- Theorem: Each sums-of-squares identity is associated with a normed algebra over the real numbers.

Normed algebras

- A *real algebra* is a vector space over $\mathbb{R}$ that has a vector multiplication that distributes over vector addition.
- A *normed algebra* is a real algebra $\mathbb{A}$ equipped with a mapping $N : \mathbb{A} \to \mathbb{R}$ such that $N(uv) = N(u)N(v)$ for all $u, v \in \mathbb{A}$.
- The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ both have such norms.
- We'll meet the other two shortly.

The $(7,3,1)$ block design

B_1	124
B_2	235
B_3	346
B_4	457
B_5	561
B_6	672
B_7	713

The $(7,3,1)$ block design is:

- a set V of 7 items, and a collection of 7 subsets of V called *blocks*, such that
- each block contains three items,
- each item is in three blocks, and
- each pair of items is in exactly one block together.

The incidence matrix for $(7, 3, 1)$

- The incidence matrix for $(7, 3, 1)$ is the 7×7 $(0, 1)$ matrix M with $M_{ij} = 1$ if and only if B_i contains the jth object:

	1	2	3	4	5	6	7	
B_1	1	1	0	1	0	0	0	124
B_2	0	1	1	0	1	0	0	235
B_3	0	0	1	1	0	1	0	346
B_4	0	0	0	1	1	0	1	457
B_5	1	0	0	0	1	1	0	561
B_6	0	1	0	0	0	1	1	672
B_7	1	0	1	0	0	0	1	713

Remember this, because it's important.

One and two squares

One square

- The identity: $a^2 \cdot b^2 = (ab)^2$
- The algebra: $\mathbb{R}$, the real numbers
- The norm: $N(r) = r^2$.

Two squares

- The identity: $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$, due to Diophantus (3rd century) and Brahmagupta (7th century)
- The algebra: $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}, i^2 = -1\}$, the complex numbers
- The norm: $N(a + bi) = a^2 + b^2$
- The origins of $\mathbb{C}$: solution of the cubic, differential equations, the geometric complex plane

Four squares

- The identity, due to Euler (1748):

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) =$$
$$= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)^2 + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)^2$$
$$+(a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2)^2 + (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)^2$$

- The algebra: the quaternions
  $\mathbb{H} = \{a_1 + a_2 i + a_3 j + a_4 k | a_1, a_2, a_3, a_4 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$
- The norm: $N(a_1 + a_2 i + a_3 j + a_4 k) = a_1^2 + a_2^2 + a_3^2 + a_4^2$
- The origins: W. R. Hamilton, who —
  - failed in an attempt to define a multiplication on $\mathbb{R}^3$, and then —
  - succeeded in defining a *noncommutative* multiplication on $\mathbb{R}^4$.

Eight squares

$$
\begin{aligned}
&(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2) \\
&\times(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2) \\
&= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 - a_5 b_5 - a_6 b_6 - a_7 b_7 - a_8 b_8)^2 \\
&+(a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 + a_5 b_6 - a_6 b_5 - a_7 b_8 + a_8 b_7)^2 \\
&+(a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 + a_5 b_7 + a_6 b_8 - a_7 b_5 - a_8 b_6)^2 \\
&+(a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 + a_5 b_8 - a_6 b_7 + a_7 b_6 - a_8 b_5)^2 \\
&+(a_1 b_5 - a_2 b_6 - a_3 b_7 - a_4 b_8 + a_5 b_1 + a_6 b_2 + a_7 b_3 + a_8 b_4)^2 \\
&+(a_1 b_6 + a_2 b_5 - a_3 b_8 + a_4 b_7 - a_5 b_2 + a_6 b_1 - a_7 b_4 + a_8 b_3)^2 \\
&+(a_1 b_7 + a_2 b_8 + a_3 b_5 - a_4 b_6 - a_5 b_3 + a_6 b_4 + a_7 b_1 - a_8 b_2)^2 \\
&+(a_1 b_8 - a_2 b_7 - a_3 b_6 + a_4 b_5 - a_5 b_4 - a_6 b_3 + a_7 b_2 + a_8 b_1)^2
\end{aligned}
$$

- Due to F. Degen (1818), J. T. Graves (1843) and A. Cayley (1845)

Eight squares, continued

- The algebra: the octonions $\mathbb{O}$, where $\mathbb{O} = \{a_0 + a_1 e_1 + \ldots + a_7 e_7 | a_0, \ldots, a_7 \in \mathbb{R}, e_t^2 = -1\}$; the e_t are called the *octonion units*
- The norm: $N(a_0 + a_1 e_1 + \cdots + a_7 e_7) = a_1^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2$
- The origins: J. T. Graves, who —
  - received Hamilton's letter announcing the quaternions, and then —
  - went him one better by defining a noncommutative and *nonassociative* multiplication on $\mathbb{R}^8$.
- The multiplication is given by the following table.

Multiplying the octonion units

$*$	$\mathbf{1}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_4}$	$\mathbf{e_5}$	$\mathbf{e_6}$	$\mathbf{e_7}$
$\mathbf{1}$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\mathbf{e_1}$	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
$\mathbf{e_2}$	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
$\mathbf{e_3}$	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
$\mathbf{e_4}$	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
$\mathbf{e_5}$	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
$\mathbf{e_6}$	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
$\mathbf{e_7}$	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Multiplication table for the octonion units.

This looks very strange . . .

The table as a matrix of signs of the e_i

... but looks are deceptive.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

The matrix, altered slightly . . .

- Remove the top row and left column, replace each -1 by a zero, and move the bottom row to the top. Here's the result:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

. . . becomes the incidence matrix for $(7, 3, 1)$

	1	2	3	4	5	6	7	
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B_7	1	0	1	0	0	0	1	713

I *told* you it was important.

Octonion multiplication, explained

- The following are the seven blocks of the $(7, 3, 1)$ design, with the given internal orderings: $(1, 2, 4)$, $(2, 3, 5)$, $(3, 4, 6)$, $(4, 5, 7)$, $(5, 6, 1)$, $(6, 7, 2)$, and $(7, 1, 3)$.
- For distinct $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$, define $e_i e_j = e_k = -e_j e_i$, where (i, j, k) is the unique block containing i and j, in the given internal ordering.
- What is $e_4 e_6$? The relevant block is $(3, 4, 6)$, so $e_4 e_6 = e_3$.
- What is $e_5 e_1$? The relevant block is $(5, 6, 1)$, so $e_5 e_1 = -e_1 e_5 = -e_6$.

That's why "the octonion units" is a name for $(7, 3, 1)$.
But wait — there's another reason.

$\mathbb{O}$ is another name for $(7,3,1)$

Seven quaternion subalgebras of $\mathbb{O}$

- $\mathbb{O}$ contains seven complex subalgebras $\mathbb{C}_n = \mathbb{R}\langle e_n \rangle$ and seven quaternion subalgebras $\mathbb{H}_n = \mathbb{R}\langle e_t, e_u, e_v \rangle$, where $\{t, u, v\}$ is a block in $(7,3,1)$
- Each $\mathbb{H}_n$ contains three of the $\mathbb{C}_k$.
- Each $\mathbb{C}_k$ is contained in three of the $\mathbb{H}_n$.
- Each pair $\{\mathbb{C}_k, \mathbb{C}_m\}$ is contained in a unique $\mathbb{H}_n$ together.

The $(7,3,1)$ design sits inside $\mathbb{O}$

- The above design of subalgebras of $\mathbb{O}$ has the same incidence matrix as $(7,3,1)$.
- The two designs are the same!

THANK YOU!