Pythagorean Triples and Generalized Fibonacci Numbers

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1. Pythagorean Triples
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Definition

A **Pythagorean triple** (PT) is an ordered triple of positive integers, (a, b, c), such that $a^2 + b^2 = c^2$. The PT is called **primitive** provided $\gcd(a, b, c) = 1$.

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Theorem

Let (a, b, c) be a PT. (a, b, c) is primitive if and only if it can be written in the form $(m^2 - n^2, 2mn, m^2 + n^2)$ where $m, n \in \mathbb{N}$, $m > n$, $gcd(m, n) = 1$, and $m + n$ is odd.

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Theorem

Let t be an even positive integer and $r, s \in \mathbb{N}$ such that $t^2 = 2rs$. Then, $(r + t, s + t, r + s + t)$ is a PT. The PT is primitive if and only if $gcd(r, s) = 1$.

WLOG, we'll assume s is even.

Forms of PTs

Example

Let $t = 12$. $t^2 = 2 \cdot 72$, so we have several choices for r and s:

$r = 1$, $s = 72$: (13,84,85)

$r = 2$, $s = 36$: (14,48,50)

$r = 3$, $s = 24$: (15,36,39)

$r = 4$, $s = 18$: (16,30,34)

$r = 6$, $s = 12$: (18,24,30)

$r = 8$, $s = 9$: (20,21,29)

Definition

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A **generalized Fibonacci sequence** $\{w_n\}$ is given by $w_0 = c, w_1 = d$ and for $n \geq 2$, $w_n = aw_{n-1} + bw_{n-2}$, where $a, b, c, d \in \mathbb{Z}$.

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Examples

$a = 1,\ b = 1,\ c = 0,\ d = 1$: $\{F_n\}$ Fibonacci seq. $0, 1, 1, 2, 3, 5, 8, \cdots$
$a = 1,\ b = 1,\ c = 2,\ d = 1$: $\{L_n\}$ Lucas seq. $2, 1, 3, 4, 7, 11, 18, \cdots$
$a = 2,\ b = 1,\ c = 0,\ d = 1$: $\{P_n\}$ Pell seq. $0, 1, 2, 5, 12, 29, 70, \cdots$
$a = 1,\ b = 2,\ c = 0,\ d = 1$: $\{J_n\}$ Jacobsthal seq. $0, 1, 1, 3, 5, 11, 21, \cdots$

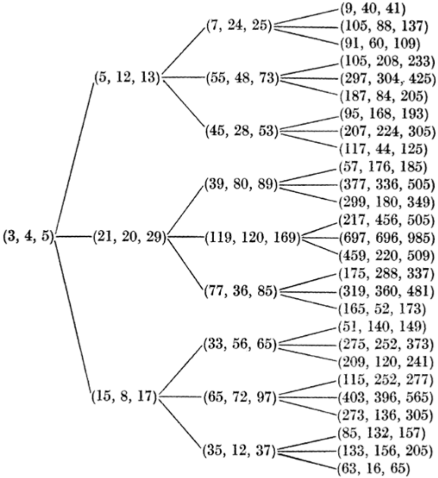
Bicknell-Johnson (1979)

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$	
2	1	4	$3 = F_4$	$5 = F_5$	
3	2	12	$5 = F_5$	$13 = F_7$	
3	1	6	$8 = F_6$	10	(not primitive)
4	1	$8 = F_6$	15	17	
7	6	84	$13 = F_7$	85	
5	2	20	$21 = F_8$	29	
11	10	220	$21 = F_8$	221	
5	3	30	16	$34 = F_9$	(not primitive)
17	1	$34 = F_9$	288	290	(not primitive)
8	3	48	$55 = F_{10}$	73	
28	27	1512	$55 = F_{10}$	1513	
8	5	80	39	$89 = F_{11}$	
45	44	3960	$F_{11} = 89$	3961	
37	35	2590	$144 = F_{12}$	2594	(not primitive)
20	16	640	$144 = F_{12}$	656	(not primitive)
15	9	270	$144 = F_{12}$	306	(not primitive)
13	5	130	$144 = F_{12}$	194	(not primitive)
9	8	$144 = F_{12}$	17	145	
72	1	$144 = F_{12}$	5183	5185	
36	2	$144 = F_{12}$	1292	1300	(not primitive)
24	3	F_{12}	567	585	(not primitive)
18	4	F_{12}	308	340	(not primitive)
12	6	F_{12}	108	180	(not primitive)
13	8	208	105	$233 = F_{13}$	
117	116	27144	$233 = F_{13}$	27145	
16	11	352	135	$377 = F_{14}$	
19	4	152	345	$377 = F_{14}$	

Bicknell-Johnson (1979)

m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$	k
F_{k+1}	F_k	$F_{k+2}F_{k-1}$	$2F_{k+1}F_k$	F_{2k+1}	$k \geq 2$
F_{k+1}	F_{k-1}	F_{2k}	$2F_{k+1}F_{k-1}$	$F_k^2 + 2F_{k-1}F_{k+1}$	$k \geq 2$
F_k	1	$F_k^2 - 1$	$2F_k$	$F_k^2 + 1$	$k \geq 3$
$\frac{F_{6k}}{2}$	1	$\frac{(F_{6k}^2 - 4)}{4}$	F_{6k}	$\frac{(F_{6k}^2 + 4)}{4}$	$k \geq 1$
$\frac{F_{3k+1} + 1}{2}$	$\frac{F_{3k+1} - 1}{2}$	F_{3k+1}	$\frac{F_{3k+1}^2 - 1}{2}$	$\frac{F_{3k+1}^2 + 1}{2}$	$k \geq 1$
$\frac{F_{3k-1} + 1}{2}$	$\frac{F_{3k-1} - 1}{2}$	F_{3k-1}	$\frac{F_{3k-1}^2 - 1}{2}$	$\frac{F_{3k-1}^2 + 1}{2}$	$k \geq 2$

Hall (1970)



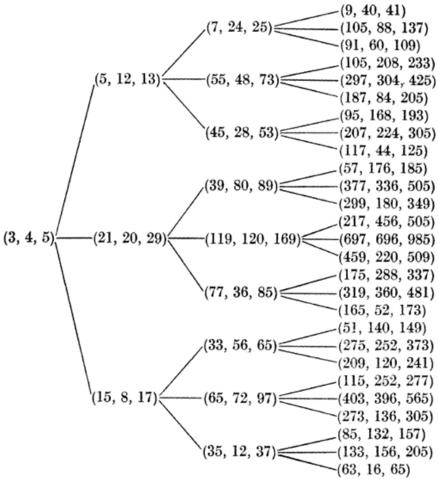
Thinking of the PTs in the tree as column vectors, you can traverse the tree using these matrices:

$$U = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

Hall (1970)



$UAU\cdots$ and $AUA\cdots$: primitive PTs where m and n are Fibonacci numbers

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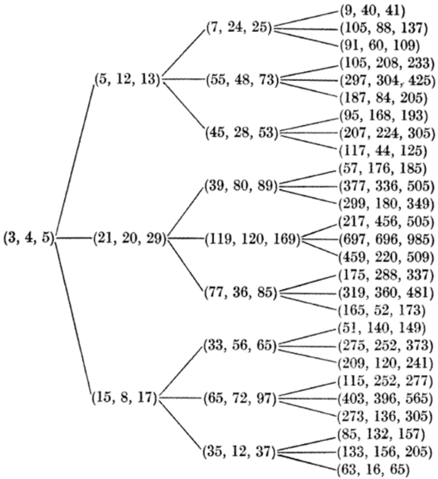
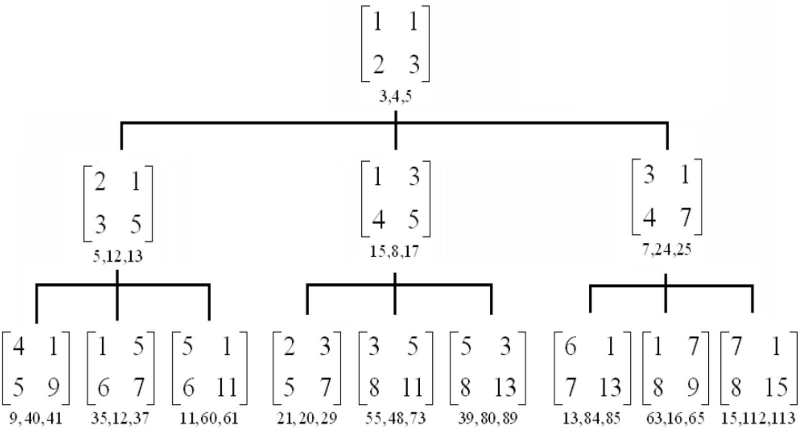


$UAU\cdots$ and $AUA\cdots$: primitive PTs where m and n are Fibonacci numbers

m	n	PT	Matrix
2	1	(3,4,5)	I
3	2	(5,12,13)	U
5	2	(21,20,29)	A
8	5	(39,80,89)	UA
8	3	(55,48,73)	AU
13	8	(105,208,233)	UAU
21	8	(377,336,505)	AUA

Price (2011)


$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ 3,4,5
$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ 5,12,13
$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ 15,8,17
$\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$ 7,24,25
$\begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix}$ 9, 40, 41
$\begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ 35,12,37
$\begin{bmatrix} 5 & 1 \\ 6 & 11 \end{bmatrix}$ 11,60,61
$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ 21, 20, 29
$\begin{bmatrix} 3 & 5 \\ 8 & 11 \end{bmatrix}$ 55,48,73
$\begin{bmatrix} 5 & 3 \\ 8 & 13 \end{bmatrix}$ 39, 80, 89
$\begin{bmatrix} 6 & 1 \\ 7 & 13 \end{bmatrix}$ 13,84,85
$\begin{bmatrix} 1 & 7 \\ 8 & 9 \end{bmatrix}$ 63,16,65
$\begin{bmatrix} 7 & 1 \\ 8 & 15 \end{bmatrix}$ 15,112,113

Price (2011)

You can also traverse Price's tree in a similar way as Hall's by treating the primitive PTs as column vectors and using the matrices

$$M_1 = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix}, M_3 = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

Powers of Hall's and Price's Matrices

Fibonacci numbers and generalized Fibonacci numbers appear in powers of some of these matrices:

Theorem

For all $n \in \mathbb{N}$,

1. $$A^n = \begin{bmatrix} \frac{P_{2n}+P_{2n-1}+(-1)^n}{2} & \frac{P_{2n}+P_{2n-1}+(-1)^{n-1}}{2} & P_{2n} \\ \frac{P_{2n}+P_{2n-1}+(-1)^{n-1}}{2} & \frac{P_{2n}+P_{2n-1}+(-1)^n}{2} & P_{2n} \\ P_{2n} & P_{2n} & P_{2n}+P_{2n-1} \end{bmatrix}$$
2. $$(UA)^n = \begin{bmatrix} \frac{1}{2}F_{3n}^2 + (-1)^n & F_{3n}^2 & \frac{1}{2}F_{6n} \\ F_{3n}^2 & 2F_{3n}^2 + (-1)^n & F_{6n} \\ \frac{1}{2}F_{6n} & F_{6n} & \frac{5}{2}F_{3n}^2 + (-1)^n \end{bmatrix}$$

Theorem

3. $$(AU)^n = \begin{bmatrix} \frac{5}{2}F_{3n}^2 + (-1)^n & -F_{6n} & \frac{3}{2}F_{6n} \\ F_{6n} & (-1)^n - 2F_{3n}^2 & 3F_{3n}^2 \\ \frac{3}{2}F_{6n} & -3F_{3n}^2 & \frac{9}{2}F_{3n}^2 + (-1)^n \end{bmatrix}$$

4. $$(DA)^n = \begin{bmatrix} 2F_{3n}^2 + (-1)^n & F_{3n}^2 & F_{6n} \\ F_{3n}^2 & \frac{1}{2}F_{3n}^2 + (-1)^n & \frac{1}{2}F_{6n} \\ F_{6n} & \frac{1}{2}F_{6n} & \frac{5}{2}F_{3n}^2 + (-1)^n \end{bmatrix}$$

5. $$(AD)^n = \begin{bmatrix} (-1)^n - 2F_{3n}^2 & F_{6n} & 3F_{3n}^2 \\ -F_{6n} & \frac{5}{2}F_{3n}^2 + (-1)^n & \frac{3}{2}F_{6n} \\ -3F_{3n}^2 & \frac{3}{2}F_{6n} & \frac{9}{2}F_{3n}^2 + (-1)^n \end{bmatrix}$$

Powers of Hall's and Price's Matrices

Theorem

For all $n \in \mathbb{N}$,

$$M_2^n = \begin{bmatrix} 2^n J_n + 4J_{n-1}^2 & (-1)^{n+1} J_n & 2^n J_n + 4J_{n-1}^2 - 1 \\ 2^n J_n & (-2)^n & 2^n J_n \\ 2^n J_n + 4J_n J_{n-1} & (-1)^n J_n & 2^n J_n + 4J_n J_{n-1} + 1 \end{bmatrix}$$

PTPMs

Definition

A **Pythagorean triple preserving matrix (PTPM)** is a 3×3 matrix that transforms any PT into another PT.

Other PTPMs with Fibonacci Numbers

Theorem

Let $B = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$. *Then for all* $n \in \mathbb{N}$,

$$B^n = \begin{bmatrix} \frac{1}{2}F_n^2 + (-1)^n & F_n^2 & \frac{1}{2}F_{2n} \\ F_n^2 & 2F_n^2 + (-1)^n & F_{2n} \\ \frac{1}{2}F_{2n} & F_{2n} & \frac{5}{2}F_n^2 + (-1)^n \end{bmatrix}$$

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Other PTPMs with Fibonacci Numbers

Example

$$\begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix} (2,1) \mapsto (3,2)$$

$$\begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix} = \begin{bmatrix} 16 \\ 30 \\ 34 \end{bmatrix} (3,2) \mapsto (5,3)$$

$$\begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 16 \\ 30 \\ 34 \end{bmatrix} = \begin{bmatrix} 39 \\ 80 \\ 89 \end{bmatrix} (5,3) \mapsto (8,5)$$

Other PTPMs with Fibonacci Numbers

Theorem

Suppose $\{w_n\}$ *is given by* $w_0 = c, w_1 = d$ *and for* $n \geq 2$, $w_n = aw_{n-1} + bw_{n-2}$, *where* $a, b, c, d \in \mathbb{Z}$. *The following matrix is a PTPM and transforms a PT generated by consecutive terms of* w_k *into the next such PT.*

$$M_{a,b} = \begin{bmatrix} \frac{a^2 - b^2 - 1}{2} & ab & \frac{a^2 + b^2 - 1}{2} \\ a & b & a \\ \frac{a^2 - b^2 + 1}{2} & ab & \frac{a^2 + b^2 + 1}{2} \end{bmatrix}$$

Example

$$\begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} -2 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

Fibonacci Pell Jacobsthal

Other PTPMs with Fibonacci Numbers

Theorem

Let $\{t_n\}$ be the sequence defined by $t_0 = 0, t_1 = 1$ and $t_n = at_{n-1} + t_{n-2}$ for $n \geq 2$. Then, for $n \geq 1$,

$$M_{a,1}^n = \begin{bmatrix} \frac{a^2}{2} t_n^2 + (-1)^n & at_n^2 & \frac{a}{2} t_{2n} \\ at_n^2 & 2t_n^2 + (-1)^n & t_{2n} \\ \frac{a}{2} t_{2n} & t_{2n} & \left(\frac{a^2}{2} + 2\right) t_n^2 + (-1)^n \end{bmatrix}.$$

Other PTPMs with Fibonacci Numbers

Theorem

Let $B = \begin{bmatrix} 6 & 2 & 6 \\ 2 & 3 & 3 \\ 6 & 3 & 7 \end{bmatrix}$. *Then, for all* $n \in \mathbb{N}$,

$$B^n = 2^{n-1} \begin{bmatrix} 4F_{2n}^2 + 2 & 2F_{2n}^2 & 2F_{4n} \\ 2F_{2n}^2 & F_{2n}^2 + 2 & F_{4n} \\ 2F_{4n} & F_{4n} & 5F_{2n}^2 + 2 \end{bmatrix}$$

Other PTPMs with Fibonacci Numbers

Theorem

Let $(r+t, s+t, r+s+t)$ be a PT in rst-form. Then, $\begin{bmatrix} r & s & t \\ s & r & t \\ t & t & r+s \end{bmatrix}$ *is a PTPM.*

Other PTPMs with Fibonacci Numbers

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Examples

$t = 2,\ r = 1,\ s = 2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$t = 12,\ r = 9,\ s = 8$

$$\begin{bmatrix} 9 & 8 & 12 \\ 8 & 9 & 12 \\ 12 & 12 & 17 \end{bmatrix}$$

Other PTPMs with Fibonacci Numbers

Theorem

Let $k \in \mathbb{N}$. Then, each of the following is a PTPM.

1. $$\begin{bmatrix} \frac{F_{3k}^2}{4} & 2 & F_{3k} \\ 2 & \frac{F_{3k}^2}{4} & F_{3k} \\ F_{3k} & F_{3k} & \frac{F_{3k}^2}{4}+2 \end{bmatrix}$$
2. $$\begin{bmatrix} 1 & \frac{F_{3k}^2}{2} & F_{3k} \\ \frac{F_{3k}^2}{2} & 1 & F_{3k} \\ F_{3k} & F_{3k} & \frac{F_{3k}^2}{2}+1 \end{bmatrix}$$
3. $$\begin{bmatrix} F_{k-1}^2 & 2F_k^2 & 2F_kF_{k-1} \\ 2F_k^2 & F_{k-1}^2 & 2F_kF_{k-1} \\ 2F_kF_{k-1} & 2F_kF_{k-1} & F_{k-1}^2+2F_k^2 \end{bmatrix}$$

Theorem

4. $$\begin{bmatrix} F_k^2 & 2F_{k-1}^2 & 2F_kF_{k-1} \\ 2F_{k-1}^2 & F_k^2 & 2F_kF_{k-1} \\ 2F_kF_{k-1} & 2F_kF_{k-1} & F_k^2+2F_{k-1}^2 \end{bmatrix}$$

5. $$\begin{bmatrix} (F_k-1)^2 & 2 & 2(F_k-1) \\ 2 & (F_k-1)^2 & 2(F_k-1) \\ 2(F_k-1) & 2(F_k-1) & (F_k-1)^2+2 \end{bmatrix}$$

6. $$\begin{bmatrix} \frac{(F_{6k}-2)^2}{4} & 2 & F_{6k}-2 \\ 2 & \frac{(F_{6k}-2)^2}{4} & F_{6k}-2 \\ F_{6k}-2 & F_{6k}-2 & \frac{(F_{6k}-2)^2}{4}+2 \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & \frac{(F_{3k+1}-1)^2}{2} & F_{3k+1}-1 \\ \frac{(F_{3k+1}-1)^2}{2} & 1 & F_{3k+1}-1 \\ F_{3k+1}-1 & F_{3k+1}-1 & \frac{(F_{3k+1}-1)^2}{2}+1 \end{bmatrix}$$

8. $$\begin{bmatrix} 1 & \frac{(F_{3k-1}-1)^2}{2} & F_{3k-1}-1 \\ \frac{(F_{3k-1}-1)^2}{2} & 1 & F_{3k-1}-1 \\ F_{3k-1}-1 & F_{3k-1}-1 & \frac{(F_{3k-1}-1)^2}{2}+1 \end{bmatrix}$$

A Few References

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Thanks!

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