Spectral Analysis of the Graph Laplacian on Homogeneous Trees

Matvey Yutin

Thomas Jefferson High School for Science and Technology

Outline

Graph Laplacian

The Graph Laplacian is the discrete analog of the Laplacian:

- $\Delta_g = A - D$
- $[\Delta_g u]_x = \sum_{y \sim x} u_y - u_x = -deg(x) u_x + \sum_{y \sim x} u_y$

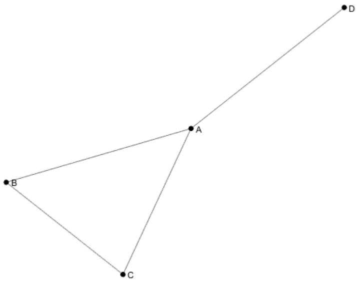

Figure: $\Delta_g = A - D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 3 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

Finite vs. Infinite Dimensions

- Spectrum of an operator Δ_g: The set of λ for which $\Delta_g - \lambda$ is not invertible.
- Finite-dimensional: Eigenvalues
- Infinite-dimensional:
  - Point Spectrum (isolated eigenvalues of finite multiplicity)
  - Continuous Spectrum (depends on the space under consideration)
- ℓ^p-space: The space of u for which $(\sum_x |u_x|^p)^{\frac{1}{p}}$ converges.
- Spectrum depends on choice of p

Infinite Homogeneous Trees

- $deg(x) = q \quad \forall x$
- $\lambda \in [-q - 2\sqrt{q-1}, -q + 2\sqrt{q-1}]$ (Lubotzky, Phillips, Sarnak)
- Main question: What, if any relationship is there between the spectrum of the infinite tree and a truncated version?

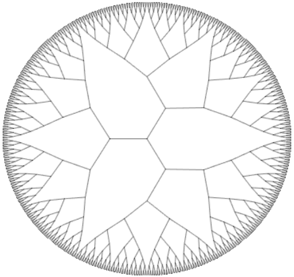

Figure: Homogeneous Tree of degree 3

Recurrence Relations: Sketch of Spectrum Proof

- Take a level-symmetric function, obtain a recurrence
- $(q-1)u_{k+1} + u_{k-1} - qu_k = \lambda u_k$
- $u_{k+1} = \frac{q+\lambda}{q-1}u_k - \frac{1}{q-1}u_{k-1}$
- Roots are $r_{\pm} = \frac{q+\lambda \pm \sqrt{(q+\lambda)^2 - 4(q-1)}}{2(q-1)}$
- ℓ^2 decay properties are equivalent to $|r_+| = |r_-|$
- Therefore λ is within the specified interval

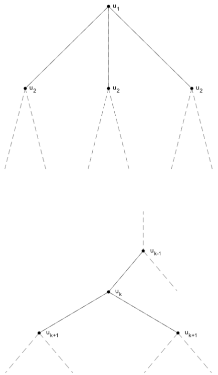

Figure: Neighborhoods of nodes in a Homogeneous Tree

Truncated Homogeneous Trees

Dirichlet: $deg(x) = q \quad \forall x$

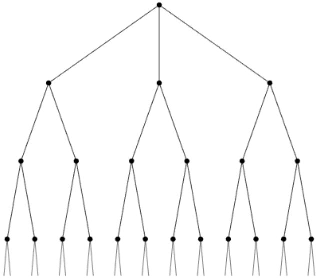

Figure: Homogeneous tree of degree 3 with 4 levels, Dirichlet truncation

Neumann: $deg(x) = q \; or \; 1 \quad \forall x$

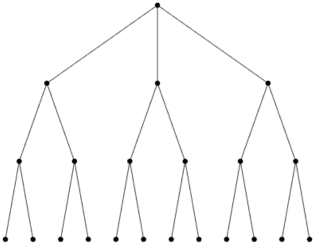

Figure: Homogeneous tree of degree 3 with 4 levels, Neumann truncation

Main Result

Theorem: The eigenvalues of the Graph Laplacian on a Dirichlet homogeneous tree of degree q are within the ℓ^2 spectrum of the Graph Laplacian on the infinite homogeneous tree of degree q

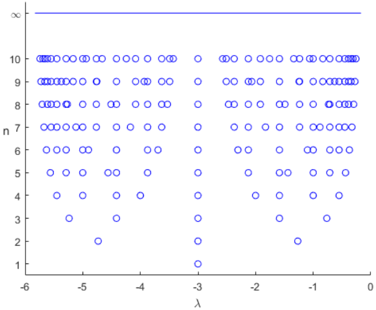

Figure: Dirichlet Eigenvalues for Homogeneous Trees of degree 3, n levels. The ℓ^2 spectrum is shown as ∞.

Finite Recurrence: Sketch of Proof

- Assume a symmetric eigenvector
- Propagate recurrence
- $r_{\pm} = \frac{q+\lambda \pm \sqrt{(q+\lambda)^2 - 4(q-1)}}{2(q-1)}$
- Solution must match boundary conditions
- Obtain a constraint on $r_{\pm}$
- Infer constraint on λ

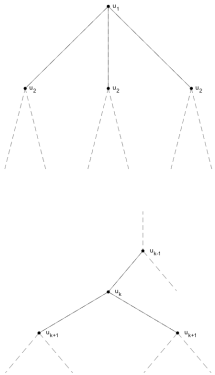

Figure: Neighborhoods of nodes in a Homogeneous Tree

Symmetric Eigenvector Structure

- Top node is 1
- All subsequent levels are symmetric

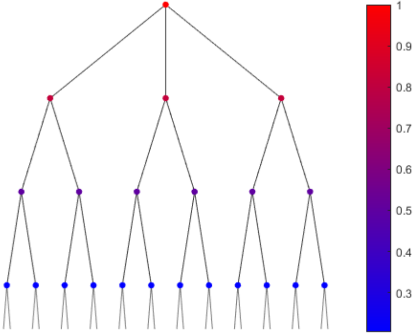

Figure: Eigenvector with eigenvalue -0.5505

Boundary Conditions

- Top Node
  - $u_1 = 1$
  - $u_2 = \frac{q+\lambda}{q}$
  - $u_k = C_+ r_+^{k-1} + C_- r_-^{k-1}$
  - $C_\pm = \pm \left(r_\pm - \frac{q+\lambda}{q(q-1)} \right) \frac{1}{r_+ - r_-}$
- Bottom Node
  - $u_{n-1} = (q+\lambda) u_n$
  - $u_{n-k} = c_+ r_+^{-k} + c_- r_-^{-k}$
  - $c_\pm = u_n \frac{\mp 1}{r_\pm} \frac{1}{(q-1)(r_+ - r_-)}$

Combining Conditions

- $\begin{cases} u_{n-1} = u_{n-1} \\ u_n = u_{n-0} \end{cases}$
- $\begin{cases} C_+ r_+^{n-2} + C_- r_-^{n-2} = c_+ r_+^{-1} + c_- r_-^{-1} \\ C_+ r_+^{n-1} + C_- r_-^{n-1} = c_+ r_+^{0} + c_- r_-^{0} \end{cases}$
- $C_+ r_+^{n-2}(r_- - r_+) = c_+ r_+^{-1}(r_- - r_+)$
- $C_\pm r_\pm^{n-1} = c_\pm$
- $\pm \left(r_\pm - \frac{q+\lambda}{q(q-1)} \right) \frac{1}{r_+ - r_-} r_\pm^{n-1} = u_n \frac{\mp 1}{r_\pm} \frac{1}{(q-1)(r_+ - r_-)}$
- $u_n = -(q-1) \left(r_\pm - \frac{q+\lambda}{q(q-1)} \right) r_\pm^n$

- WLOG assume $|r_+| > |r_-|$
- $\left| (q-1) r_+^n \left(r_+ - \frac{q+\lambda}{q(q-1)} \right) \right| = \left| (q-1) r_-^n \left(r_- - \frac{q+\lambda}{q(q-1)} \right) \right|$
- $\left| r_+ - \frac{q+\lambda}{q(q-1)} \right| < \left| r_- - \frac{q+\lambda}{q(q-1)} \right|$
- $\left| \frac{q+\lambda}{q(q-1)} \right| > \left| \frac{r_+ + r_-}{2} \right| = \left| \frac{q+\lambda}{2(q-1)} \right|$
- Contradiction, therefore $|r_+| = |r_-|$

Antisymmetric Eigenvector Structure

- First few levels are 0
- At some level, two siblings are opposite
- All their descendants are symmetric

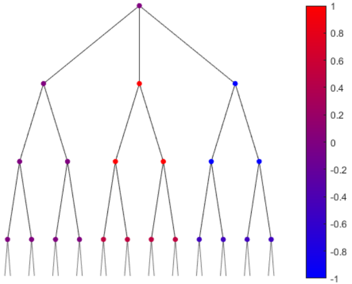

Figure: Eigenvector with eigenvalue -1

Boundary Conditions

- Top Node
  - $u_1 = 1$, note[1]
  - $u_2 = \frac{q+\lambda}{q-1}$
  - $u_k = C_+ r_+^{k-1} + C_- r_-^{k-1}$
  - $C_\pm = \pm \frac{r_\pm}{r_+ - r_-}$
- Bottom Node: same as last time
  - $u_{n-1} = (q+\lambda) u_n$
  - $u_{n-k} = c_+ r_+^{-k} + c_- r_-^{-k}$
  - $c_\pm = u_n \frac{\mp 1}{r_\pm} \frac{1}{(q-1)(r_+ - r_-)}$

[1] u_1 is taken to be the value at the first nonzero node, instead of at the root

Combining Conditions

- $\begin{cases} u_{n-1} = u_{n-1} \\ u_n = u_{n-0} \end{cases}$
- $\begin{cases} C_+ r_+^{n-2} + C_- r_-^{n-2} = c_+ r_+^{-1} + c_- r_-^{-1} \\ C_+ r_+^{n-1} + C_- r_-^{n-1} = c_+ r_+^{0} + c_- r_-^{0} \end{cases}$
- $C_+ r_+^{n-2}(r_- - r_+) = c_+ r_+^{-1}(r_- - r_+)$
- $C_\pm r_\pm^{n-1} = c_\pm$
- $\pm r_\pm \frac{1}{r_+ - r_-} r_\pm^{n-1} = u_n \frac{\mp 1}{r_\pm} \frac{1}{(q-1)(r_+ - r_-)}$
- $u_n = -(q-1) r_\pm^{n+1}$
- $(q-1) r_+^{n+1} = (q-1) r_-^{n+1}$
- $|r_+| = |r_-|$

Sufficiency Proof: Decomposition

- Repeat at each level:
  - Swap any two sibling branches below the specified level
  - Average the original and swapped
  - Do this for all pairs of branches below the specified level
  - If the specified level is the first one, you have a symmetric eigenvector, which you subtract out
  - Otherwise, swap any two sibling branches at the specified level
  - Take the difference from the step before; now you have an antisymmetric eigenvector
- This process terminates, and results in a set of component eigenvectors for the original
- Therefore every vector is a sum of symmetric and antisymmetric components

Numerical Observations: Eigenvalue Structure

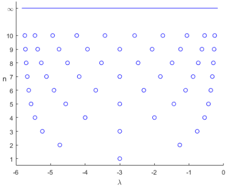

Figure: Symmetric Eigenvalues for degree 3, n levels

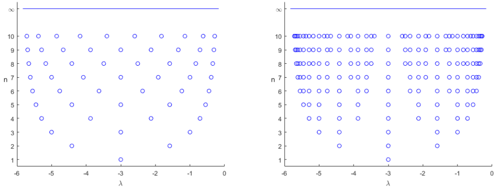

Figure: Asymmetric Eigenvalues for degree 3, new at n^{th} level and n levels respectively

Future Work

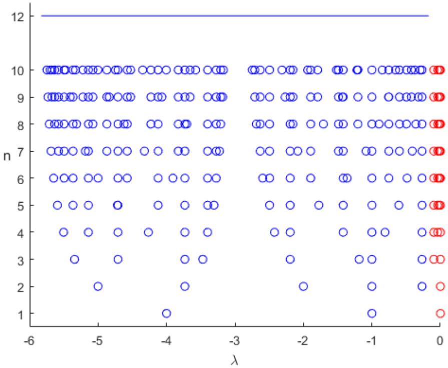

Figure: Neumann Eigenvalues for degree 3; in red outside the spectrum

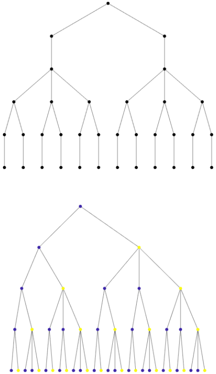

Figure: Periodic Tree and Tree of Finite Cone Type, respectively

Acknowledgments

- Dr. Matt Holzer
- George Mason University
- TJHSST Mentorship Program

References

- A. Lubotzky, R. Phillips and P. Sarnak, 'Ramanujan graphs', Combinatorica 8 (1988) 261-277.