The I Road to Higher Mathematics – Promoting Inquiry as Part of Mathematical Maturity

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- Most importantly, introducing the square root of -1 is problematic at best, so there is a need for definitions and proofs right away.

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- They are quoted the Fundamental Theorem of Algebra, so that is a main target for investigation.
- The complex squaring function and its “inverse” the complex square root can be done nicely as an application of polar coordinates. That takes care of degree 2.

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- The idea of the mapping $z \mapsto z^k$ comes into play in both, in particular the winding of the image of a circle in the range.
- The underlying notion of complex polynomials as being orientation preserving is central.

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- Power series and Fourier series are wonderful topics which can be introduced. Complex linear algebra also if students have some background.

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- Preparation for Topology: Working with definitions; continuity; winding number; topological degree.

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- the ability to write and speak about mathematics using precise mathematical language;
- an understanding of the role of definitions in mathematics and being able to use (and possibly construct) them effectively;
- a basic understanding of elementary logical principles and proof techniques.

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- the ability to create visual images from written mathematics and vice versa;
- the ability to identify similarities and differences between mathematical objects.
- knowing how to capture the essential elements of intuitive mathematical objects in precise language that can make them subject to rigorous mathematical analysis (e.g. definitions and axiom systems) and understanding the importance of this process in mathematical discourse.

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- The connections across mathematics are naturally present.
- The mathematics is inherently beautiful and surprisingly deep yet accessible.

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- Noyce Fellow (Undergrad) will be part of research into effects.

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Investigations by use of analogy – rational numbers $\mapsto$ rational functions: how are they related? different?

Gaussian integers – what are primes? What does factoring Gaussian integers tell us? Which prime integers can factor over the Gaussian integers? What if we adjoin a square root of -3 instead of -1?

Inquiry opportunites – continued

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Can consider the Cauchy "stealth function"

$$e^{-\frac{1}{x^2}}$$

and its complexified version with an essential singularity!

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Many more

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What will I do without my best material in history of math?