Combinatorial Games: Open Problems and Computational Approaches

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Hood College

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Combinatorial Games

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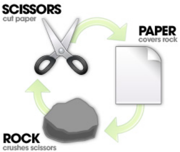

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Let's play a game!

Game of Chomp

Rules for Chomp

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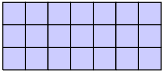

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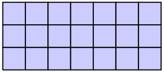

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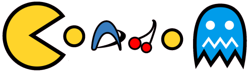

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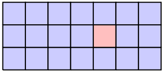

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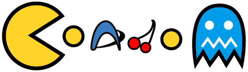

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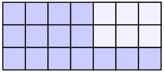

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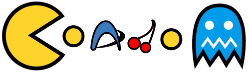

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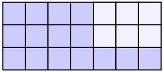

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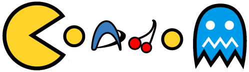

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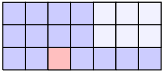

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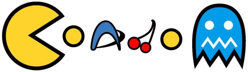

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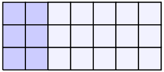

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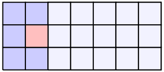

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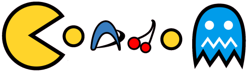

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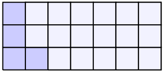

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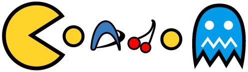

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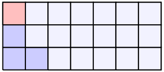

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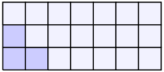

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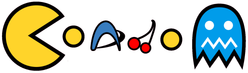

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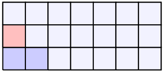

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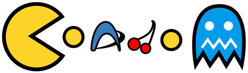

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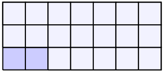

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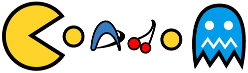

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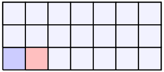

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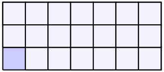

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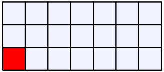

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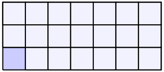

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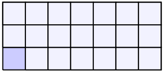

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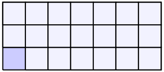

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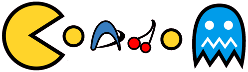

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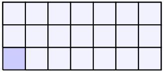

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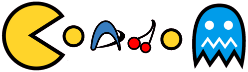

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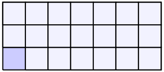

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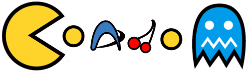

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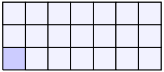

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Goal: Find solutions to games.

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- **Weak:** There is an algorithm to *describe* the optimal moves to secure a win or draw for the first (or second) player from the initial position of the game.
- **Strong:** There is an algorithm to secure a win for a player from *any* game position, even if sub-optimal moves were made in previous play.

Poset Games

Definition of Posets

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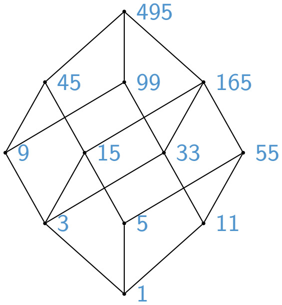

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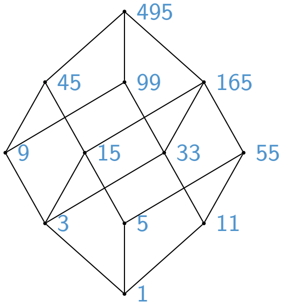
Poset Games

Given a non-empty poset P, players alternate turns selecting an element $x \in P$ and removing x and all elements y with $x \preceq y$. The loser takes the last element remaining in P.



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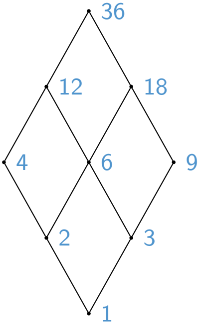
Given a positive integer n, the *game of divisors* on n is the poset game of the divisor poset D_n.



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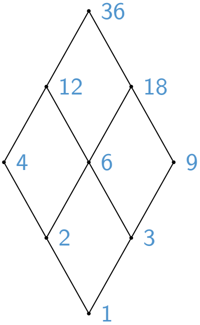

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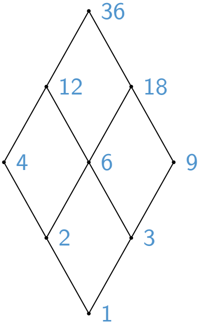

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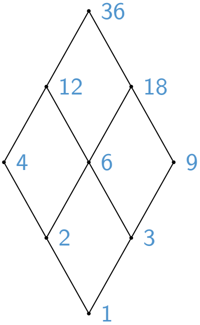

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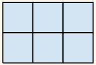

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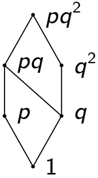

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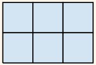

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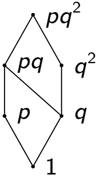

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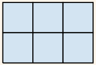

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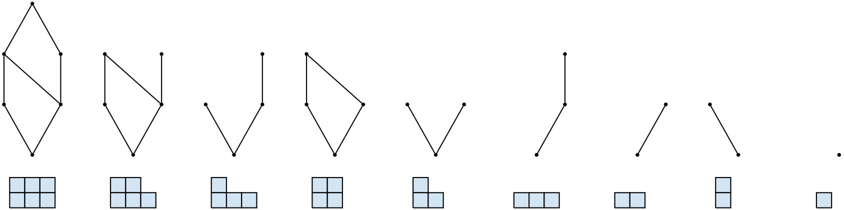
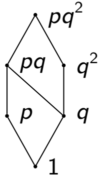
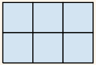


Chomp: A Special Case of the Game of Divisors



Chomp as a Game of Divisors

An $n \times m$ Chomp game is equivalent to the poset game on $D(p^{n-1} \cdot q^{m-1})$. For example, consider:



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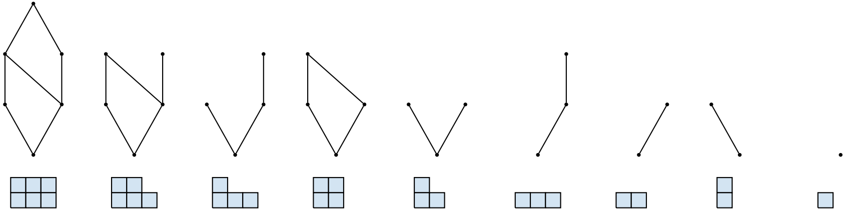
One board of game for each *antichain* of the poset.



Sprague-Grundy Function for Games

Sprague-Grundy Function

Let $\mathcal{B}_L$ be the set of boards in game G for which a player has no permissible moves (losing boards.)

- Set $SG(B) = 0$ for each board $B \in B_L$.
- Define $SG(B) = \text{mex}(\mathcal{M}_B)$, recursively for each $B \in \mathcal{B}$.

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  - Solve $D(p_1 p_2 \cdots p_k)$.

MAA MD-VA-DC Section Meeting: Fall 2015

MAA Section Meeting, Fall 2015

Thanks for listening!

Code from this talk is available at Github and on my website at:

`https://github.com/gwynwhieldon/PosetGames`

`http://cs.hood.edu/~whieldon`