Octonions, Quaternions, and Involutions

Nathaniel Schwartz
Joint work with John Hutchens

Washington College
nschwartz2@washcoll.edu

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Contents

Bilinear & Quadratic Forms

Let V be a finite dimensional vector space over a field k. Then a **bilinear form** is a mapping

$$\langle \, , \, \rangle : V \times V \to k$$

that is linear in both coordinates. We say that $\langle \, , \, \rangle$ is **nondegenerate** if

$$V^{\perp} = \{x \in V \mid \langle x, y \rangle = 0 \;\forall\; y \in V\} = 0.$$

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A **quadratic form** is a mapping

$$\mathrm{q} : V \to k$$

such that $\mathrm{q}(\lambda x) = \lambda^2 \, \mathrm{q}(x)$ for all $x \in V$, $\lambda \in k$. This uniquely determines a bilinear form

$$\langle x, y\rangle = \mathrm{q}(x+y) - \mathrm{q}(x) - \mathrm{q}(y).$$

Quadratic Forms

Notice that

$$\begin{aligned}\langle x, x\rangle &= \mathrm{q}(x+x) - \mathrm{q}(x) - \mathrm{q}(x)\\ &= \mathrm{q}(2x) - 2\,\mathrm{q}(x)\\ &= 4\,\mathrm{q}(x) - 2\,\mathrm{q}(x)\\ &= 2\,\mathrm{q}(x)\end{aligned}$$

When the characteristic is not 2, the associated symmetric bilinear form defines the quadratic form. If k has characteristic 2, then $\langle x, x\rangle = 0$ for all $x \in V$. Thus $\langle\ ,\ \rangle$ is alternate and symmetric, and the quadratic form cannot be recovered from the bilinear form.

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A vector x is called **isotropic** if $\mathrm{q}(x) = 0$, and **anisotropic** otherwise.

Algebras

An **algebra** A over a field k is a vector space equipped with a multiplication which is **not necessarily** associative:

$$x(yz) \neq (xy)z$$

or commutative:

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Examples:

- The complex numbers over (Complex multiplication).
- Euclidean 3-space (Cross product).
- Lie algebras (Lie bracket).
- Jordan Algebras (Jordan multiplication).

Jordan algebras are commutative.

Composition Algebras

A **composition algebra** C is an algebra over a field together with a quadratic form which admits composition:

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Notice that, so far, there has been no restriction made on the dimension of a composition algebra.

Doubling

Let C be a composition algebra, and let D be a finite-dimensional subalgebra. Then D is a nonsingular subspace, so $C = D \oplus D^{\perp}$, and $D^{\perp}$ is also nonsingular.

Lemma

If D is a finite-dimensional proper subalgebra of C, then there exists $a \in D^{\perp}$ so that $q(a) \neq 0$, then $D_1 = D \oplus Da$ is a composition subalgebra.

Note: the quadratic form, product and conjugation on D_1 require attention.

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Note: the quadratic form, product and conjugation on D_1 require attention.

Also notice that Da and D have the same dimension, so that D_1 has twice the dimension of D. In other words, we've just doubled D.

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Lemma

Let D be a composition algebra, and let $\lambda \in k^*$, and let $C = D \oplus D$.

1. *If D is associative, then C is a composition algebra.*
2. *C is associative if and only if D is commutative and associative.*

Differences in Low Dimension

Lemma

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In case $\mathrm{char}(k) = 2$, take some $\langle a, e \rangle \neq 0$. Then a 2-dimensional composition algebra is $ke \oplus ka$.

In either case, we get a 2-dimensional composition algebra.

Structure Theorem

Theorem

Every composition algebra is obtained by repeated doubling, starting from ke *if* $\mathrm{char}(k) \neq 2$ *or from the 2-dimensional algebra if* $\mathrm{char}(k) = 2$. *In this way, we obtain algebras of dimensions 1 (if* $\mathrm{char}(k) \neq 0$*), 2, 4, and 8. So we have a sequence*

$$D_1 \subset D_2 \subset D_3.$$

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Using the lemmas, and since D_1 is commutative and associative, we have that D_2 is associative. However, D_2 is not commutative.

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So D_3 is not associative, and thus no algebra properly contains D_3. So we are done.

Types of Compositions Algebras

A composition algebra C is called **split** if q is isotropic. If q is anisotropic, then the composition algebra is called **division**. Split algebras over the same field are isomorphic.

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It is known that all quaternion and octonion algebras are split when taken over a perfect field k of characteristic 2.

Automorphism Group

Let C be an octonion algebra. The automorphisms of C that fix a quaternion subalgebra D form a group isomorphic to G_2. That is,

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Springer & Veldkamp show that every automorphism $g \in \mathrm{Aut}(C)$ has the form

$$g(x + yu) = cxc^{-1} + (pcyc^{-1})u,$$

where $w \in D^{\perp}$, $\mathrm{q}(w) \neq 0$, $\mathrm{q}(p) = 1$, and $\mathrm{q}(c) \neq 0$. Here x, y, c and $p \in D$ and $u \in D^{\perp}$.

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Also, every automorphism of G_2 is inner.

Some Preliminary Results

Theorem

Let C be an octonion algebra. If k is a perfect field of characteristic 2, there is one isomorphism class of inner k-involutions of $\mathrm{Aut}(C)$. In this case, the fixed-point group of an inner k-involution is isomorphic to

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Question: Are there always elements of order 2 in a division quaternion algebra?

Thank you!

Questions?