Sonobe Origami for enriching understanding of geometric concepts in three dimensions

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"Programs that take advantage of paper folding to teach mathematics are thriving in many parts of the world," according to the organizers of the MAA origami-themed Contributed Paper Session to be held at the JMM in January 2016. But, K-12 should not be having all the fun! In this talk, I will show some ways of stimulating student engagement using sonobe origami. These activities can be used in General Education mathematics "appreciation" courses or for Non-Euclidean Geometry for mathematics majors. Specifically, the goals are enriching student understanding of surface curvatures and helping them understand the duality of the regular polyhedra using these folded paper objects.
Things to chat about...

• Construction overview and classroom hints
• Duality of regular polyhedra
  (aka Platonic Solids)
• Euler's $\chi$ for topological classification
• Angle deficit for topological classification
• Gauss-Bonnet Theorem
• Bonus Puzzle!
Construction Basics

- tinyurl.com/sonobe
- I also have videos on YouTube
- for Gen Ed course- one class period
- students spend about an hour outside of class
- 30 pieces of paper per student
- chirality compatibility is a construction issue
- assembly is a learning experience for students
The Paper I Use in Class:

Neenah AstroBrights Note Cube, 3.5 X 3.5 Inches, Assorted, 550 Count (20400)
by Neenah

Price: $11.07 Prime | FREE Same-Day
Delivered today for FREE with qualifying orders over $35.

In Stock.
Ships from and sold by Amazon.com. Gift-wrap available.

Want it TODAY, Oct. 29? Order within 8 hrs 50 mins and choose Same-Day Delivery at checkout.

- Acid free for great quality that doesn't deteriorate
- 11 Bright colors
- Perfect for any home or office
- Perfect to keep by the phone
- 550 Sheets

Roll over image to zoom in
Pros and Cons

- No glue
- No scissors
- Straightedge and Compass is equivalent to folding
- Pride in crafting

- Only time for one large project rather than many small ones
- Object is more complex than regular polyhedra
Make 30 of those
Hyperbolic vertex forms here

Pyramid forms here
Ok, Great!

Now, what is this sonobe project good for?
Duality of Regular Polyhedra

CUBE

OCTAHEDRON

TETRAHEDRON

ICOSAHEDRON

DODECAHEDRON
Cube with Octahedron

behance.net
Icosahedron with Dodecahedron
Tetrahedron is self dual
How much paper is that!?

To build the basic object, consider the underlying icosahedron (20 sides) and count pyramids.

Each side has a pyramid attached which uses 3 units. But each unit is used in two pyramids.

\[ 20 \times \frac{3}{2} = 30 \]
Other objects

- Tetrahedron: $4 \times \frac{3}{2} = 6$ units, but it's a cube!
- Octahedron: $8 \times \frac{3}{2} = 12$ units
- Icosahedron: $20 \times \frac{3}{2} = 30$ units
- Massive ball: $(12 \times 5) \times \frac{3}{2} = 90$ units

All of these objects are based off of an underlying triangular mesh frame.
Euler's $\chi$ for topological classification
Euler's $\chi$ for topological classification

- 6 vertices = $v$
- 9 edges = $e$
- 4 faces = $f$
- $\chi = v - e + f$
- $\chi = 1$

for single components with no holes
Euler's $\chi$ for topological classification

- 16 vertices = $v$
- 28 edges = $e$
- 14 faces = $f$
- 2 "blobs" = $b$
- $\chi = v - e + f - b$

$\chi = (16+14)-(28+2) = 0$

for "donuts"
Euler's $\chi$ for topological classification

- 8 vertices = $v$
- 12 edges = $e$
- 6 faces = $f$
- 1 “blob” = $b$

$\chi = v - e + f - b$

$\chi = 8 - 12 + 6 - 1 = 1$

for “spheres”
Euler's χ for topological classification

- 6+8 vertices = v
- (3+3/2)*8 edges = e
- 8*3 faces = f
- 1 “blob” = b
- χ = v – e + f - b
- χ = 14 - 36 + 24 – 1 = 1

for “spheres”

mathcraft.wonderhowto.com
Euler's $\chi$ for topological classification

- 20 + 12 vertices = $v$
- $(3+3/2)\times20$ edges = $e$
- $20\times3$ faces = $f$
- 1 “blob” = $b$
- $\chi = v - e + f - b$
- $\chi = 32 - 90 + 60 - 1 = 1$
  for “spheres”
Angle deficit for topological classification
Angle deficit for topological classification

“And now for something completely different…”
Angle deficit for topological classification

- Imagine a flattened cube
- At each vertex, 90 degrees is “missing”
- $8 \times 90 = 720$
- Total Angle Deficit is 720
Angle deficit for topological classification

- A flattened cube!
- At each vertex, 90 degrees is “missing”
- $8 \times 90 = 720$
- Total Angle Deficit is 720
Cuboctahedron Template

60° missing
Cuboctahedron in 3D

- 12 vertices
- 60 degree deficit per
- $12 \times 60 = 720$
- Total Angle Deficit is 720
Sonobe!

- 20 vertices with 90 degree deficit (elliptic curvature!)
- 12 vertices with -90 degree deficit (hyperbolic curvature)
- $20 \times 90 - 12 \times 90 = 720$
But....

Total Angle Deficit = \(4\pi \times \chi\)
Gauss-Bonnet Theorem
Gauss-Bonnet Theorem

If we trace out a closed curve on a surface, the total enclosed Gauss curvature (total angle deficit) is $2\pi$ minus the total angle defect (angles of deflection) around the curve.
Gauss-Bonnet simple example

- Total angle defect is zero. (This paper ribbon is a geodesic.)
- Each half of the sphere must have the same enclosed Gauss curvature.
  - Total must be $4\pi$
  - Each half contains $2\pi$
Gauss-Bonnet simple example

- Total angle defect is zero. (This paper ribbon is a geodesic.)
- Each half of the can must have the same enclosed Gauss curvature.
- Total must be $4\pi$
- Each half contains $2\pi$
Gauss-Bonnet simple example

- Total angle defect is zero.
- Each half of the cube must have the same enclosed Gauss curvature.
- Each half contains $2\pi$
- Each vertex has $\pi/2$
Gauss-Bonnet example

- Total angle defect is: 
  \[2(180-72)+2(180-120) = 336\]
- 360-336=24 degrees
- Angle deficit: 
  \[360-(60+60+108+108) = 24 \text{ degrees}\]
Gauss-Bonnet example

- Total angle defect is: 
  \[3(180-84)\]
  288
- \(360-288=72\) degrees
- Angle deficit: 
  \[3(360-60-60-108-108)\]
  720 degrees
Gauss-Bonnet example

- Total angle defect is: \[5(180-120)\]
  \[300\]

- \[360-300 = 60 \text{ degrees}\]

- Angle deficit:
  \[5(360-108-120-120)\]
  \[60 \text{ degrees}\]
Gauss-Bonnet example

- Total angle defect is zero (geodesic!)
- 360 degrees is promised
- Angle deficit: 5(90)+(-90) 360 degrees
Bonus Puzzle

Use 3 colors, 10 pieces of each color.

Try to make sure you get exactly one of each color on each pyramid.

(Solve using a graph first.)
Bonus Puzzle Solution
THIS ONE

Math 154
Great Ideas

Math 221
Calc 1

Math 211
Applied Calc 1

Math 170
PreCalc

Math 222
Calc 2