Cannonballs, Triangles, and Secrets

An introduction to elliptic curve cryptography

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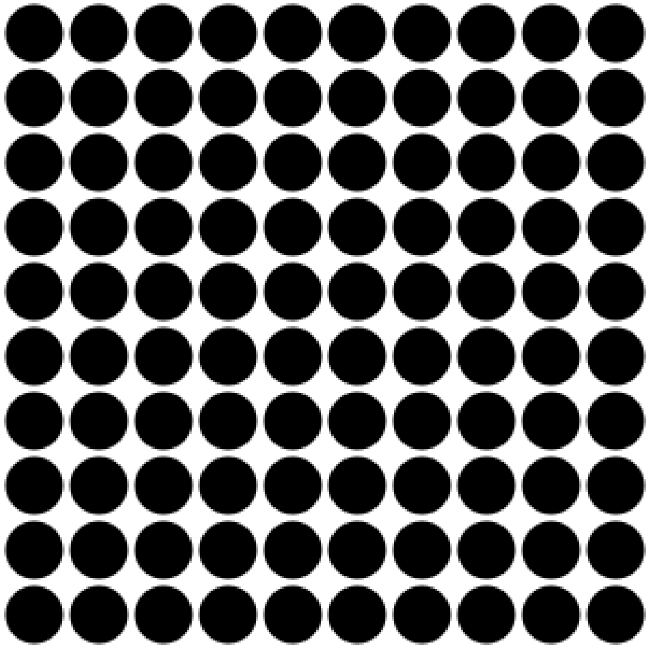
The Mathematical Association of America
Maryland-District of Columbia-Virginia Section

A Pile of Cannonballs



A Square of Cannonballs

1

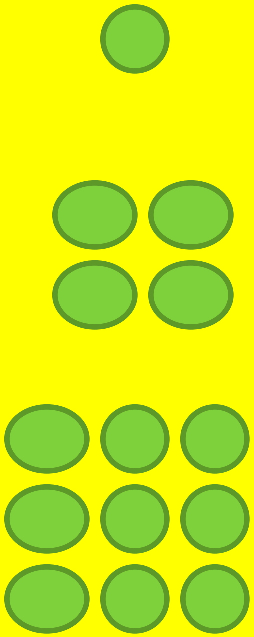
4

9

.

.

.



The number of cannonballs in x layers is

$$1 + 4 + 9 + \ldots + x^2$$

$$= x\,(x + 1)\,(2x + 1)/6$$

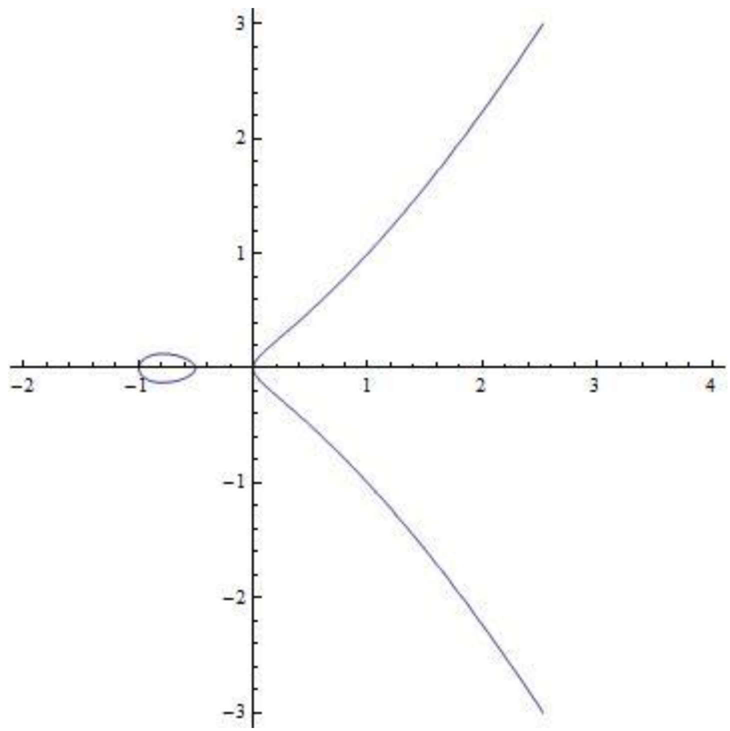
$x=3$:

$$1 + 4 + 9 = 3(4)(7)/6 = 14$$

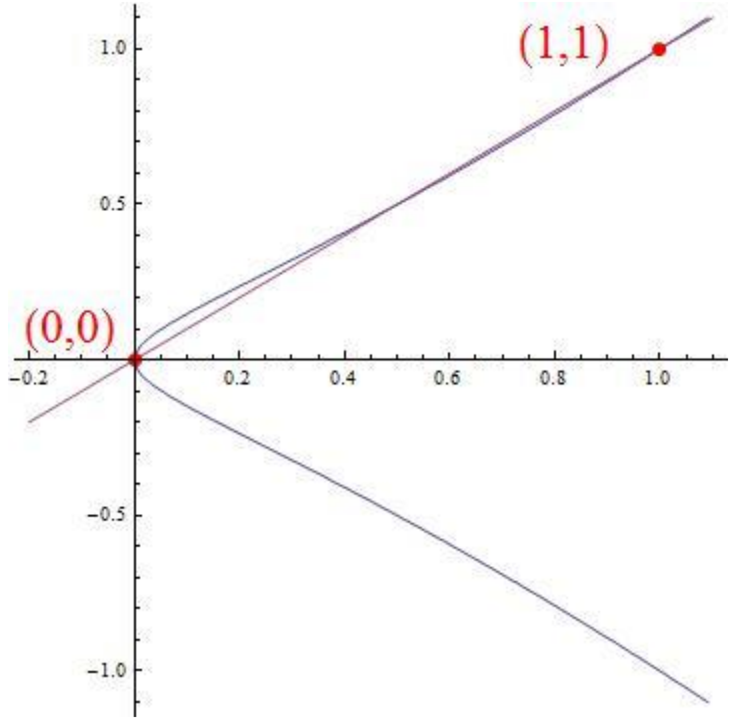
If x layers of the pyramid yield a y by y square, we need

$$y^2 = 1 + 4 + 9 + . . . + x^2$$

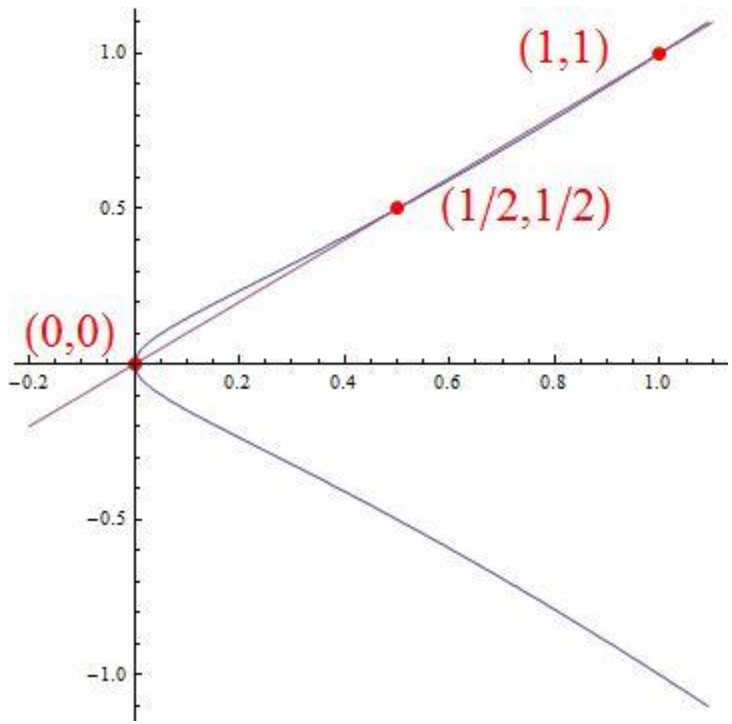
$$y^2 = x (x + 1) (2x + 1)/6$$



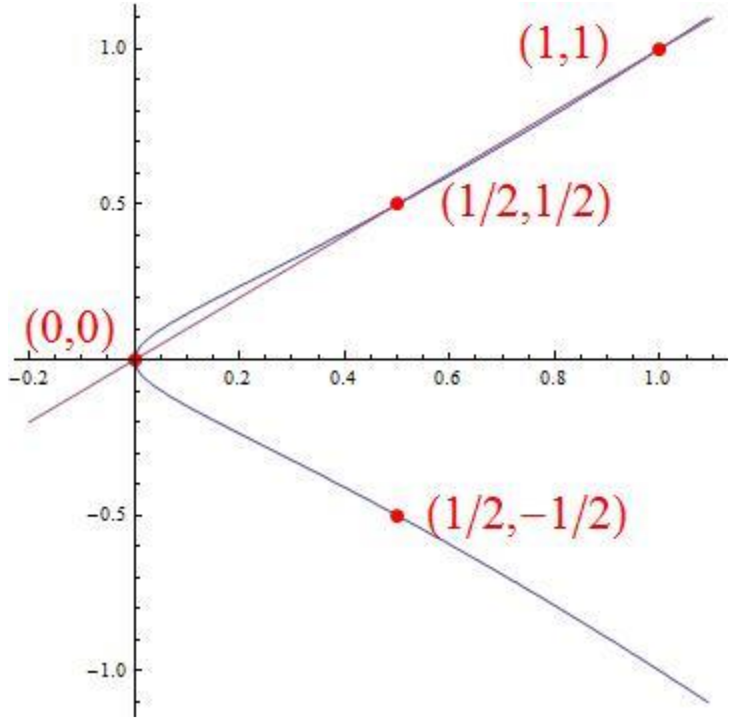
$$y^2 = x\,(x + 1)\,(2x + 1)/6$$

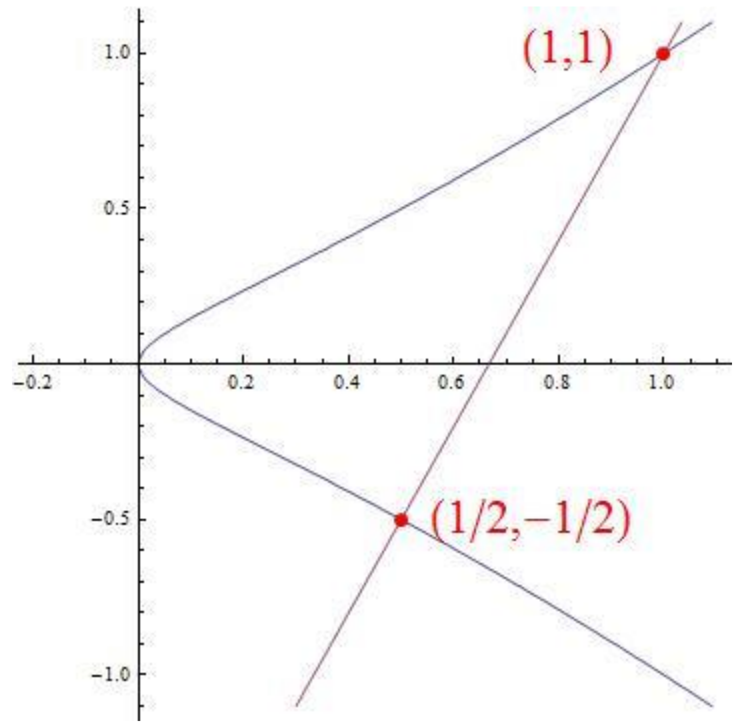


(1,1)
(0,0)
1.0
0.5
−0.5
−1.0
−0.2
0.2
0.4
0.6
0.8
1.0



$$y^2 = x\,(x + 1)\,(2x + 1)/6 \quad \text{and} \quad y = x$$


(1,1)
(1/2,1/2)
(0,0)
(1/2,−1/2)
1.0
0.5
−0.5
−1.0
−0.2
0.2
0.4
0.6
0.8
1.0



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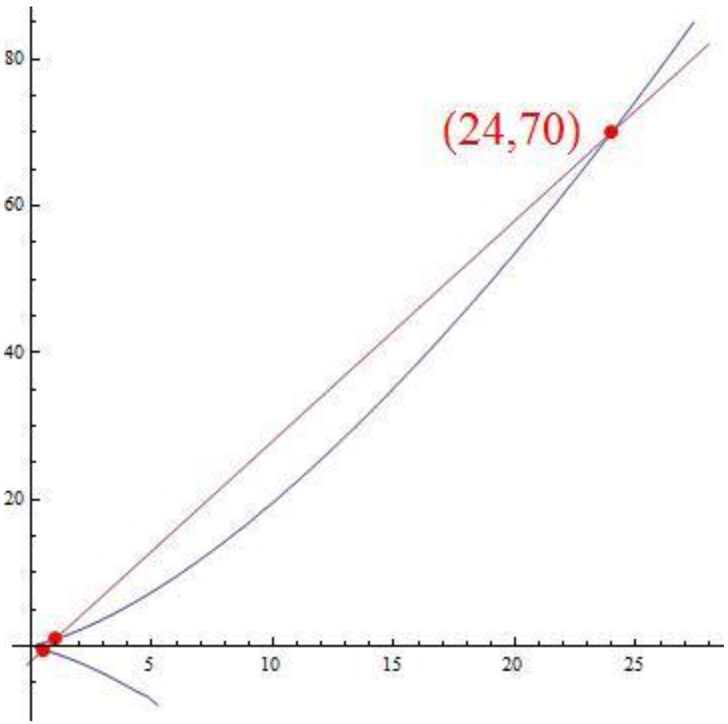
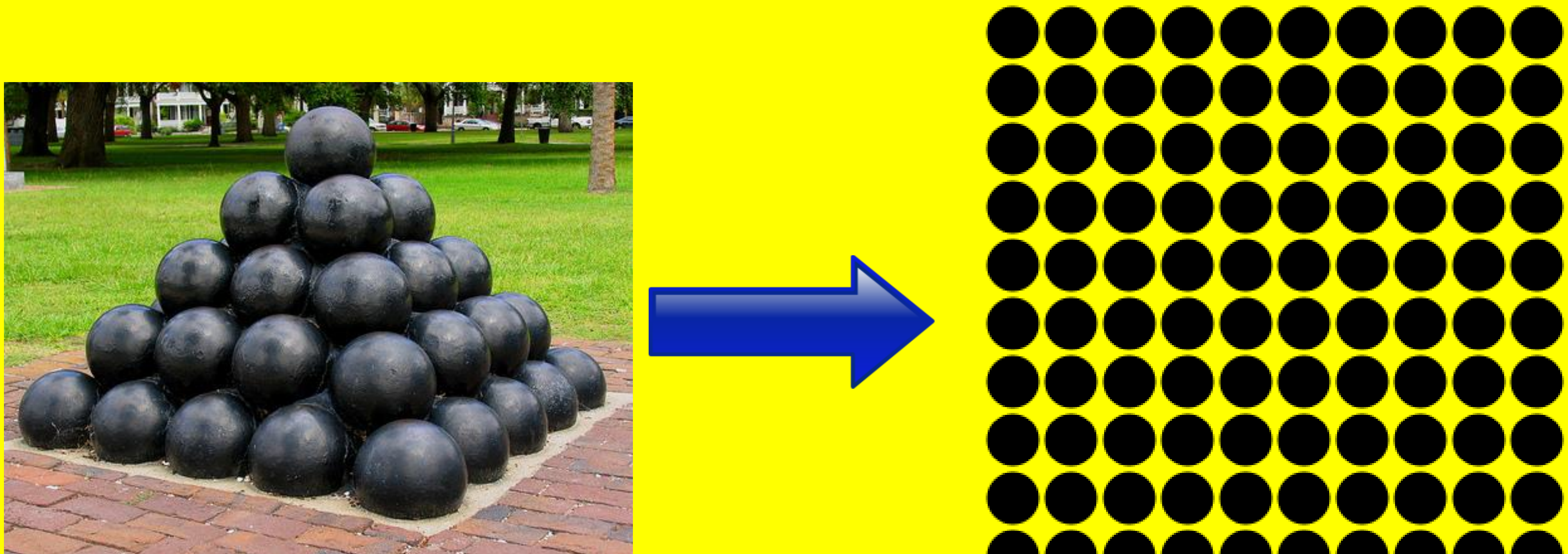
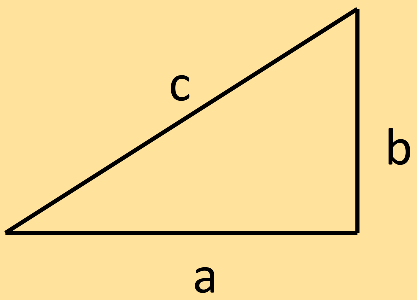


80
60
40
20
5
10
15
20
25
(24,70)

$$1 + 4 + 9 + . . . + 24^2 = 70^2$$



Is there a right triangle with rational sides whose area is 5 ?



$$a^2 + b^2 = c^2$$

$$ab/2 = 5$$

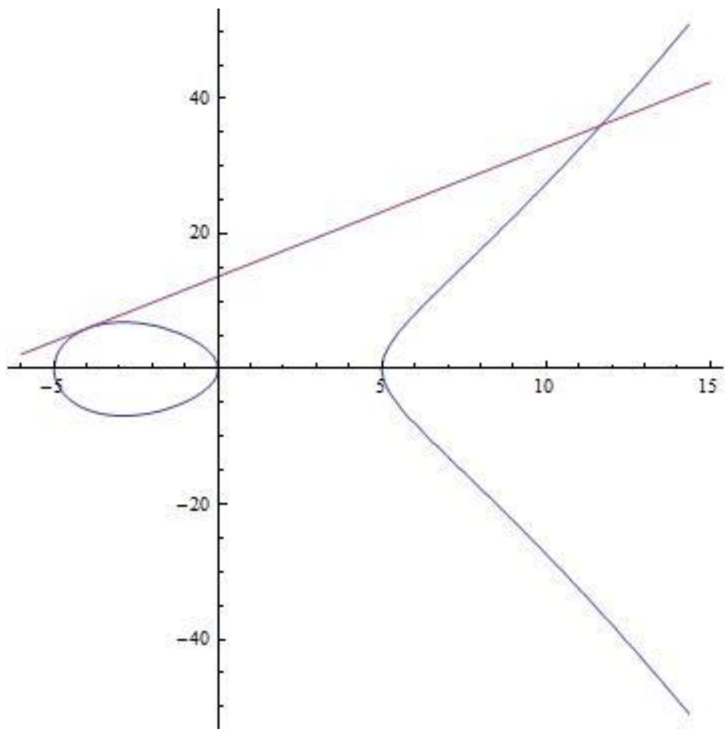
If we have a, b, c, let $x = \frac{1}{4} c^2$

Then $x - 5 = \frac{1}{4}(c^2 - 2ab) = \frac{1}{4}(a^2 + b^2 - 2ab) = \frac{1}{4}(a-b)^2$

Similarly, $x + 5 = \frac{1}{4}(a+b)^2$

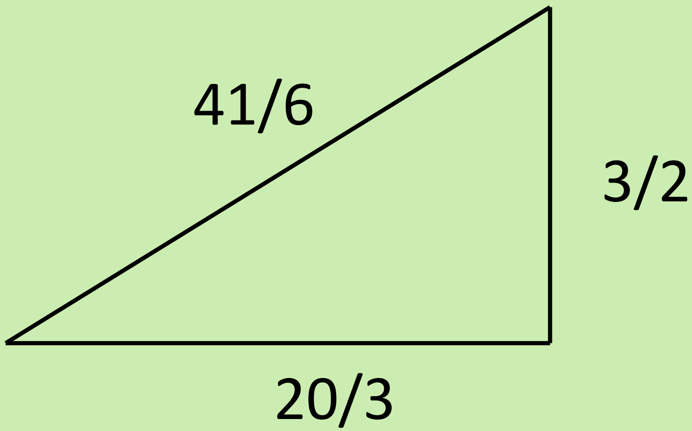
Therefore, $x^3 - 25x = x(x-5)(x+5)$ is a square.

We need points on the curve $y^2 = x^3 - 25x$ with rational coordinates.



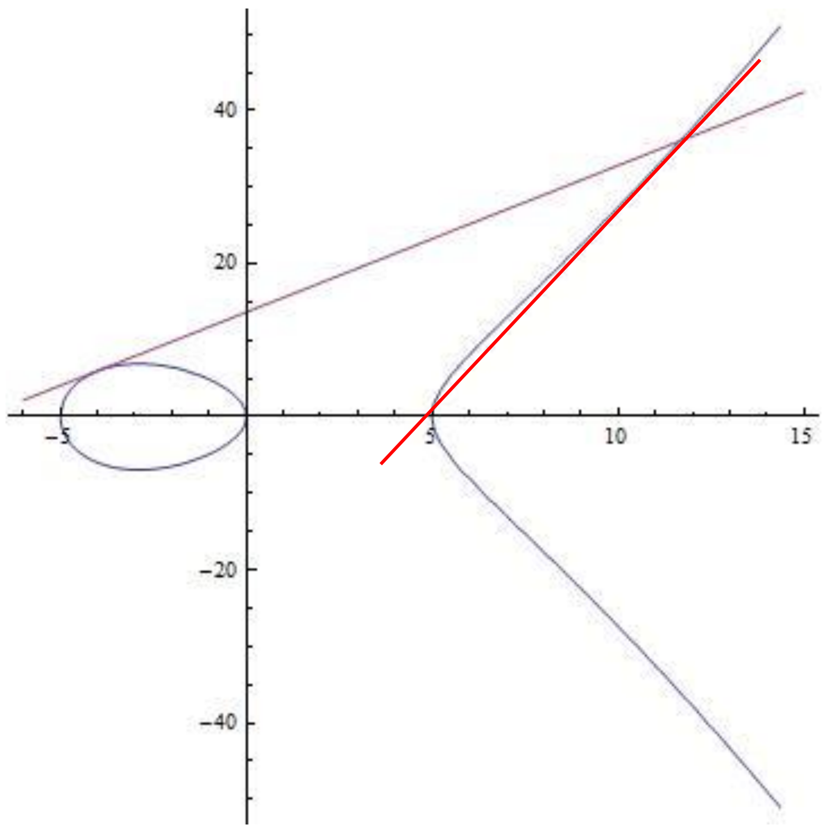
$(x, y) = (-4, 6)$ is a point on the curve. Draw the tangent line at this point.

The intersection point has $x = 1681/144 = (41/12)^2$

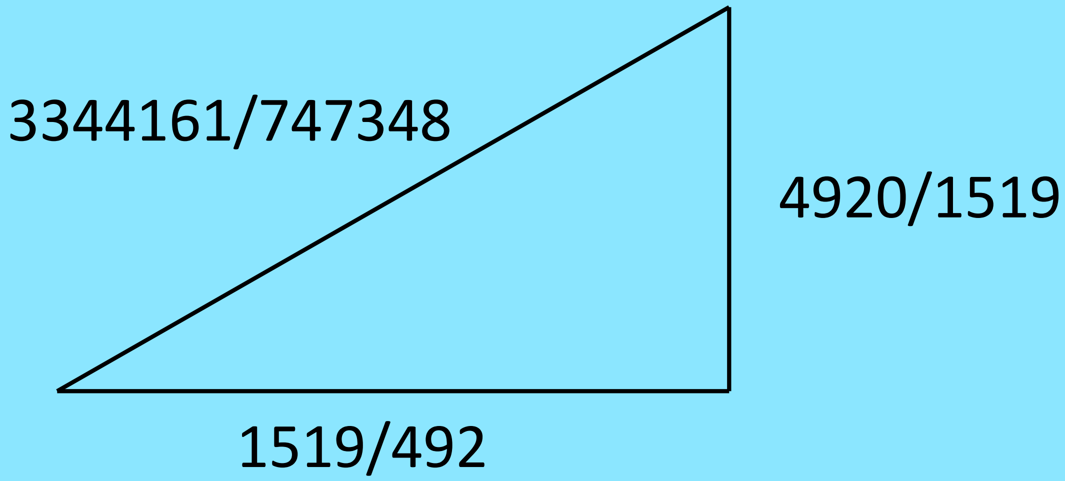


Area = 5

We can use the tangent line at this new point and find another triangle:



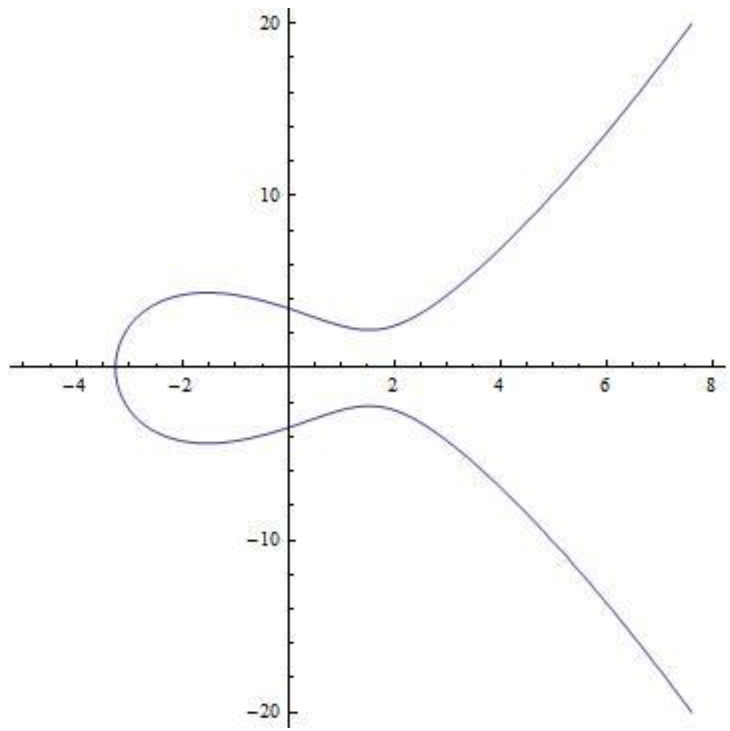
$x = (3344161/1494696)^2$, y = a big fraction



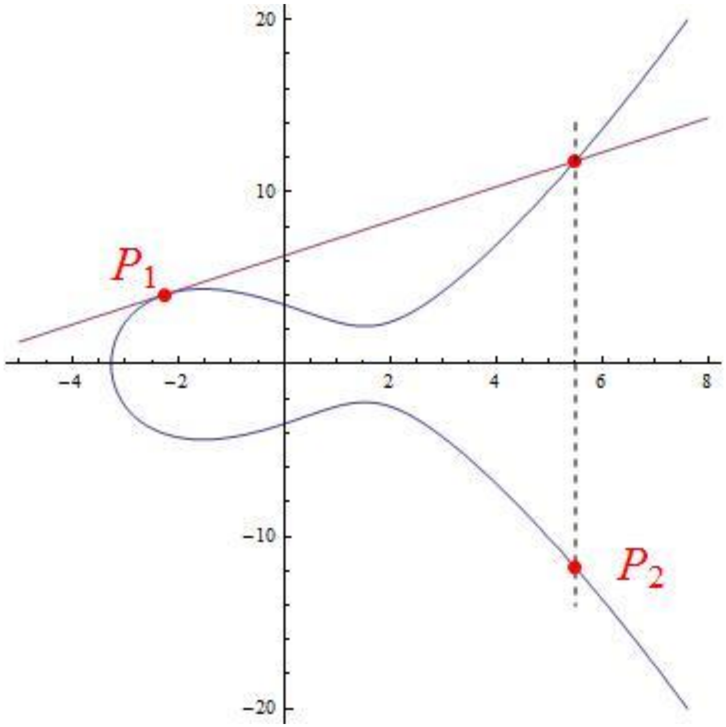
We could produce many more . . .
But soon the whole page would not be large enough to contain the numbers.

An elliptic curve is the graph of an equation

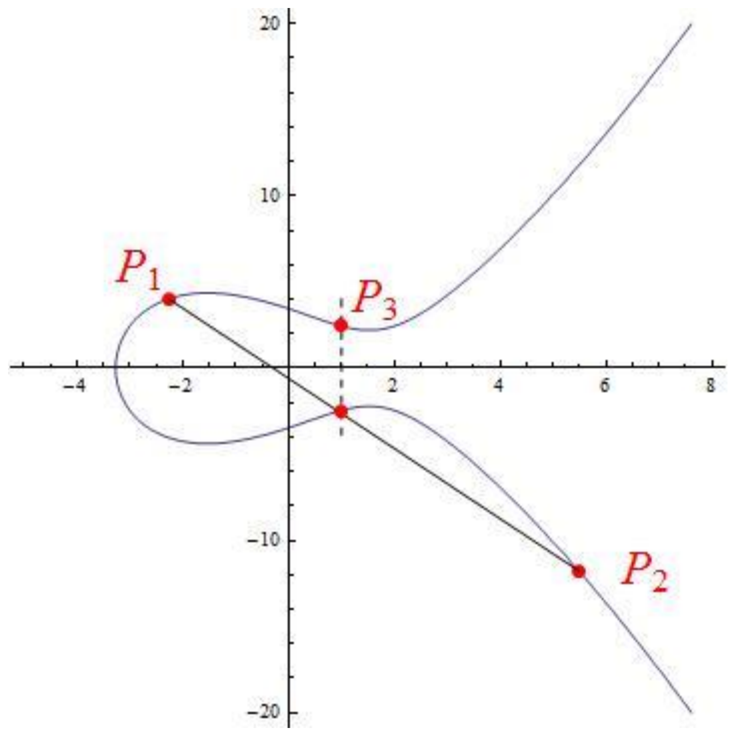
y^2 = cubic polynomial in x



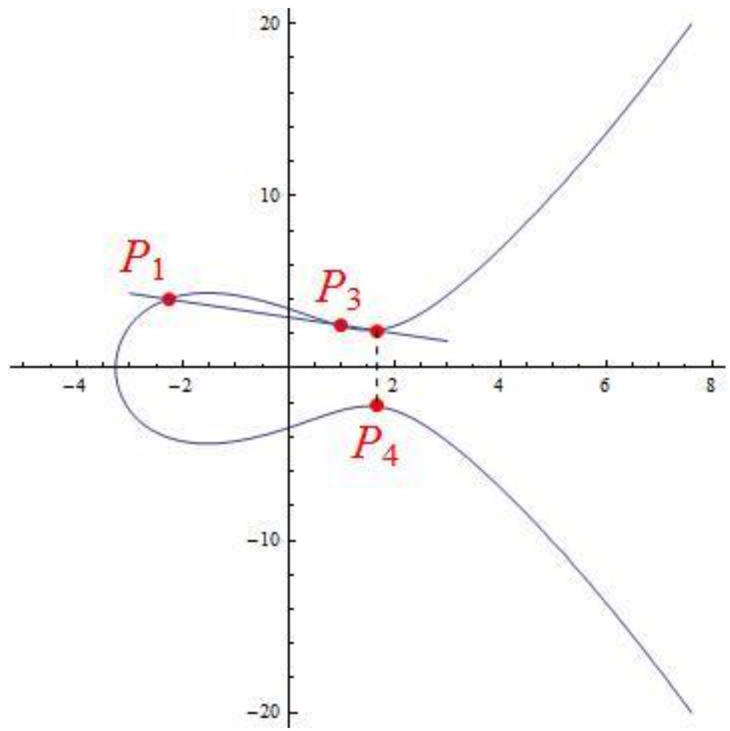
For example, $y^2 = x^3 - 5x + 12$



Start with P_1. We get P_2.



Using P_1 and P_2, we get P_3.



Using P_1 and P_3, we get P_4.

We get points P_1, P_2, P_3, . . . , P_n , . . .

Useful facts:

If we take the line through P_m and P_n and reflect the third point of intersection across the y-axis, we get P_{n+m}

If we start with P_1 , after m steps we get P_m
If we start with P_m , after n steps we get P_{mn}

All of these calculations are done mod a big prime.
Otherwise, the computer overflows.

Given n, it is easy to compute P_n
(even when n is a 1000-digit number)

Given P_n, it is very difficult to figure out the value of n.

Is this good for anything?

There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world.

—Nikolai Lobachevsky (1792-1856)

“Do you know the secret?”

The Eavesdropper

The secret is a 200-digit integer s.
Prove to me that you know the secret.

I send you a random point P_1.

You compute P_s and send it back to me.

If your answer is correct, I decide that you know the secret.

Diffie – Hellman Key Establishment

Alice and Bob want to agree on a key for use in a cryptosystem.

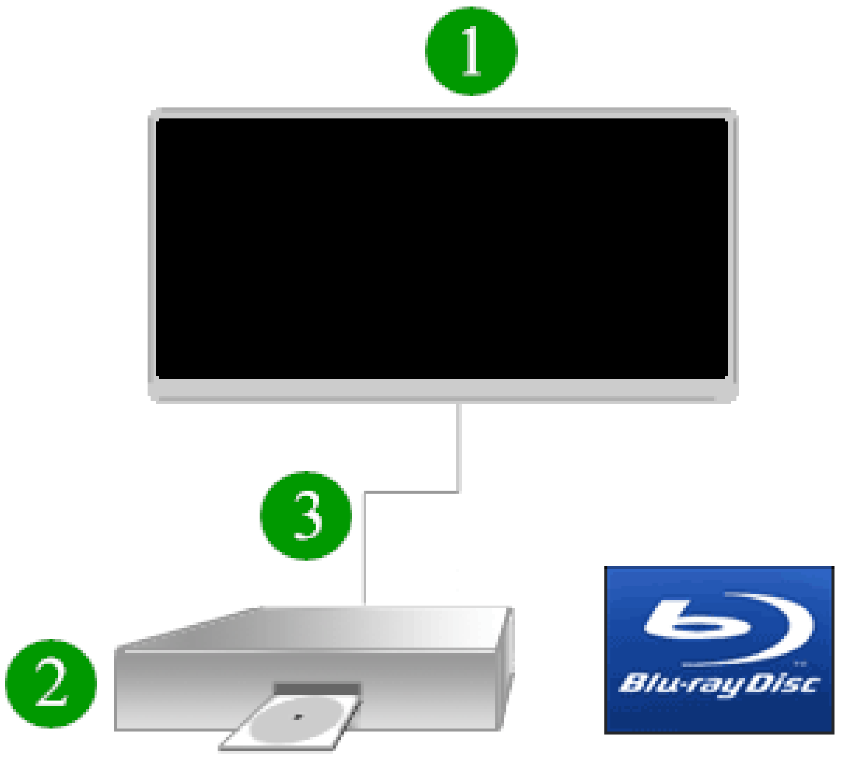
1. They choose an elliptic curve and a point P_1 on the curve.

2. Alice chooses a secret integer a and Bob chooses a secret integer b.

3. Alice computes P_a and Bob computes P_b. They exchange P_a and P_b.

4. Alice does a steps starting with P_b and computes P_{ba} , and Bob computes P_{ab}

5. They use the coordinates of P_{ab} to construct the desired key.


1
3
2
Blu-ray Disc


GUITAR HERO
Wii
PAL
DOLBY


BlackBerry
12:21
WED, MAY 25
EDGE
(3) Messages
DEL
SYM
SPACE

DIOPHANTUS

Lived from ?? to ??

Probably about 1800 years ago.

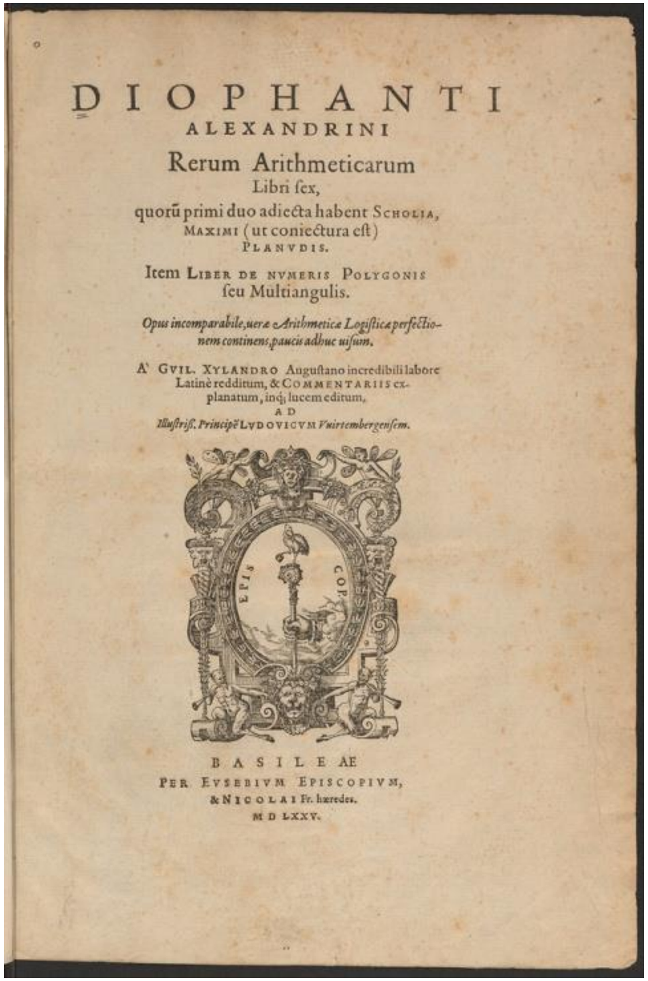
Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh as a bachelor. Five years after his marriage was born a son who died four years before his father, at half his father's age.

How many years did Diophantus live?

Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh as a bachelor. Five years after his marriage was born a son who died four years before his father, at half his father's age.

How many years did Diophantus live?

84


DIOPHANTI
ALEXANDRINI
Rerum Arithmeticarum
Libri ſex,
quorũ primi duo adiecta habent SCHOLIA,
MAXIMI (ut coniectura eſt)
PLANVDIS.
Item LIBER DE NVMERIS POLYGONIS
ſeu Multiangulis.
Opus incomparabile, ueræ Arithmeticæ Logiſticæ perfectionem continens, paucis adhuc uiſum.
A' GVIL. XYLANDRO Auguſtano incredibili labore Latinè redditum, & COMMENTARIIS explanatum, inq; lucem editum,
AD
Illuſtriſ. Principẽ LVDOVICVM *Vuirtembergenſem.*

BASILEAE
PER EVSEBIVM EPISCOPIVM,
& NICOLAI Fr. hæredes.
M D LXXV.

Problem 1

To divide a given number into two having given difference.

Given number 100,
Given difference 40.

Lesser number is x. Larger is x + 40.
Therefore

$$2x+40 = 100$$
$$x = 30$$

The required numbers are 70, 30.

$$K^{Y}\beta \pitchfork \Delta^{Y}\gamma\,\iota\sigma M\delta$$

$$2x^3 - 3x^2 = 4$$

$$s\alpha\iota\sigma M\beta$$

$$x = 2$$

ιβ ιζ	ϕιβ α	S^xλε
$\frac{17}{12}$	$\frac{1}{512}$	$\frac{35}{x}$

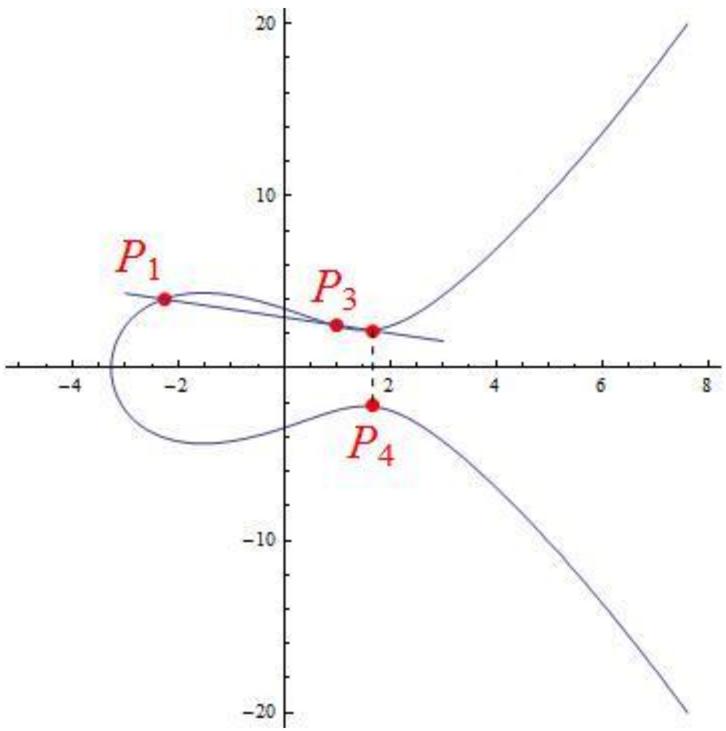
Δ^{Y}ιε ⋔ Μλς εν μοριω Δ^{Y}ΔαΜλς ⋔ Δ^{Y}ιβ

$(15\ x^2 - 36)/(x^4 + 36 - 12\ x^2)$

+, -	1489
=	1557
x	1620
.	1600
>	1600
a2, a3, a4, …	1634
aa, a^3, a^4, …	1637 (Descartes)

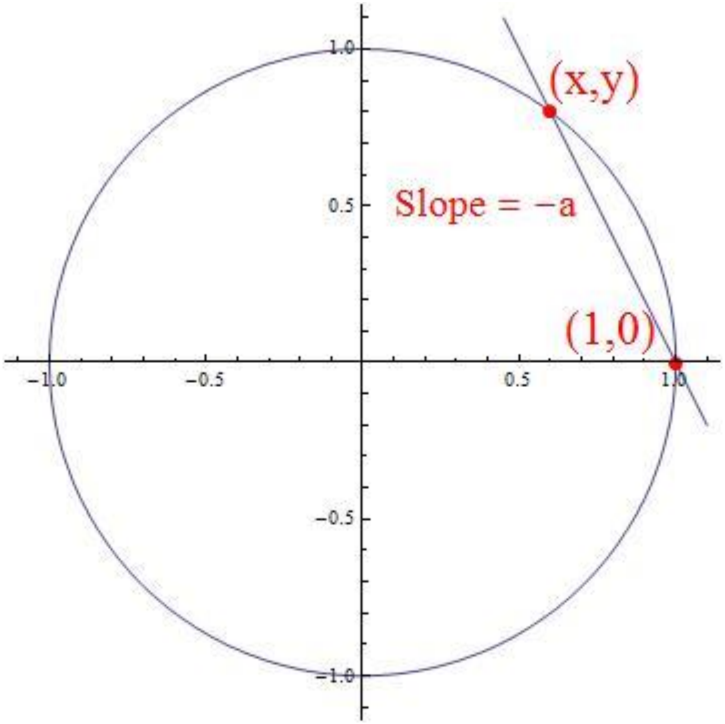
Diophantus’s goal:
Given a solution of an equation,
find another solution.

Given a point on a curve,
find another point.



Using P_1 and P_3, we get P_4.

Problem: Find rational x and y such that $x^2 + y^2 = 1$.

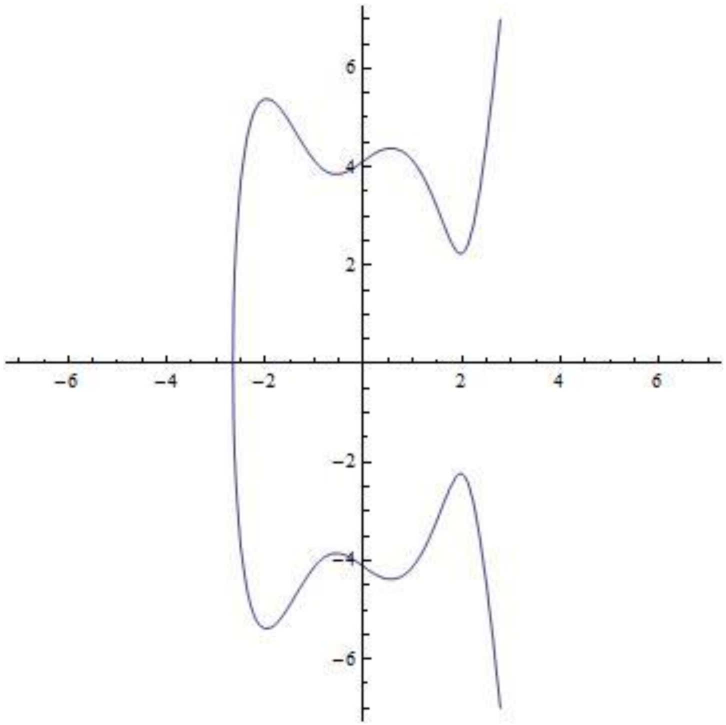


We know the easy solution x=1, y=0.

Draw the line through this point with slope, say, -2.

The second point of intersection gives a new point, in this case x= 3/5, y=4/5.

What happens if we try higher degree curves?



$y^2 = x^5 - 7x^3 + 6x + 17$

Faltings's Theorem (1983):

A higher degree curve (technically: genus > 1) has only a finite number of points with rational coordinates.

THANK YOU