Modified Picard Iteration Applied to Boundary Value Problems and Volterra Integral Equations.

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INTRODUCTION

In this talk we present a new algorithm for approximating solutions of two-point boundary value problems and provide a theorem that gives conditions under which it is guaranteed to succeed. We show how to make the algorithm computationally efficient and demonstrate how the full method works both when guaranteed to do so and more broadly. In the first section the original idea and its application are presented. We also show how to modify the same basic idea and procedure in order to apply it to problems with boundary conditions of mixed type. In the second section we introduce a new algorithm for the case if the original algorithm failed to converge on a long interval. We split the long interval into subintervals and show the new algorithm gives convergence to the solution. Finally, we repose a Volterra equation using auxiliary variables according to Parker-Sochacki in such a way that the solution can be approximated by the Modified Picard iteration scheme.

The algorithm that will be developed in here for the solution of certain boundary value problems is based on a kind of Picard iteration as just described. It is well known that in many instances Picard iteration performs poorly in actual practice, even for relatively simple differential equations. A great improvement can sometimes be had by exploiting ideas originated by G. Parker and J. Sochacki [7]. The Parker-Sochacki method (PSM) is an algorithm for solving systems of ordinary differential equations (ODEs). The method produces Maclaurin series solutions to systems of differential equations, with the coefficients in either algebraic or numerical form. The Parker-Sochacki method convert the initial value problem to a system that the right hand side is polynomials, consequently the right hand side then are continuous and satisfy the Lipschitz condition on an open region containing the initial point $t = 0$. Moreover, the system guarantees that the iterative integral is easy to integrate. We begin with the following example, the initial value problem

$$y' = \sin y, \quad y(0) = \pi/2. \tag{1}$$

To approximate it by means of Picard iteration, from the integral equation that is equivalent to (1), namely,

$$y(t) = \frac{\pi}{2} + \int_0^t \sin y(s)\, ds$$

we define the Picard iterates

$$y_0(t) \equiv \frac{\pi}{2}, \quad y_{k+1}(t) = \frac{\pi}{2} + \int_0^t \sin y_k(s)\, ds \quad \text{for } k \geq 0.$$

The first few iterates are

$$\begin{aligned} y_0(t) &= \frac{\pi}{2} \\ y_1(t) &= \frac{\pi}{2} + t \\ y_2(t) &= \frac{\pi}{2} - \cos(\frac{\pi}{2} + t) \\ y_3(t) &= \frac{\pi}{2} + \int_0^t \cos(\sin s)\, ds, \end{aligned}$$

the last of which cannot be expressed in closed form. We define variables u and v by $u = \sin y$ and $v = \cos y$ so that $y' = u$, $u' = (\cos y)y' = uv$, and $v' = (-\sin y)y' = -u^2$, hence the original problem is embedded as the first component in the three-dimensional problem

$$\begin{aligned} y' &= u & y(0) &= \frac{\pi}{2} \\ u' &= uv & u(0) &= 1 \\ v' &= -u^2 & v(0) &= 0 \end{aligned}$$

Although the dimension has increased, now the right hand sides are all polynomial functions so quadratures can be done easily.

In particular, the Picard iterates are now

$$\mathbf{y}_0(t) = \begin{pmatrix} \frac{\pi}{2} \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{y}_{k+1}(t) = \mathbf{y}_0 + \int_0^t \mathbf{y}_k(s)\,ds = \begin{pmatrix} \frac{\pi}{2} \\ 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} u_k(s) \\ u_k(s)v_k(s) \\ -u_k^2(s) \end{pmatrix} ds.$$

The first component of the first few iterates are

$$\begin{aligned} y_0(t) &= \tfrac{\pi}{2} \\ y_1(t) &= \tfrac{\pi}{2} + t \\ y_2(t) &= \tfrac{\pi}{2} + t \\ y_3(t) &= \tfrac{\pi}{2} + t - \tfrac{1}{6}t^3 \\ y_4(t) &= \tfrac{\pi}{2} + t - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^5 \\ y_5(t) &= \tfrac{\pi}{2} + t - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^5 \\ y_6(t) &= \tfrac{\pi}{2} + t - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^5 - \tfrac{61}{5040}t^7 \end{aligned}$$

The exact solution to the problem (1) is

$$y(t) = 2\arctan(e^t)$$

the first nine terms of its Maclaurin series are

$$\tfrac{\pi}{2} + t - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^5 - \tfrac{61}{5040}t^7 + O(t^9)$$

As we can see the system of ODE proposed by Parker and Sochacki is generating a Maclaurin series solution for problem (1).

An efficient Algorithm for Approximating Solutions of Two-Point Boundary Value Problems

Consider a two-point boundary value problem of the form

$$y'' = f(t, y, y'), \qquad y(a) = \alpha, \quad y(b) = \beta. \tag{2}$$

If f is locally Lipschitz in the last two variables then by the Picard-Lindelöf Theorem, for any $\gamma \in \mathbb{R}$ the initial value problem

$$y'' = f(t, y, y'), \qquad y(a) = \alpha, \quad y'(a) = \gamma \tag{3}$$

will have a unique solution on some interval about $t = a$. Introducing the variable $u = y'$ we obtain the first order system that is equivalent to (3),

$$\begin{aligned} y' &= u \\ u' &= f(t, y, u) \\ y(a) = \alpha, \qquad u(a) &= \gamma \end{aligned} \tag{4}$$

The equivalent integral equation to (3) will be

$$y(t) = \alpha + \gamma(t-a) + \int_a^t (t-s) f(s, y(s), y'(s))\, ds, \tag{5}$$

which we then solve, when evaluated at $t = b$, for γ:

$$\gamma = \tfrac{1}{b-a} \left(\beta - \alpha - \int_a^b (b-s) f(s, y(s), y'(s))\, ds \right) \tag{6}$$

The key idea is we use picard iteration to obtain successive approximations to the value of γ. Thus the iterates are

$$\begin{aligned} y^{[0]}(t) &\equiv \alpha \\ u^{[0]}(t) &\equiv \tfrac{\beta-\alpha}{b-a} \\ \gamma^{[0]} &\equiv \tfrac{\beta-\alpha}{b-a} \end{aligned} \tag{7a}$$

and

$$\begin{aligned} y^{[k+1]}(t) &= \alpha + \int_a^t u^{[k]}(s)\, ds \\ u^{[k+1]}(t) &= \gamma^{[k]} + \int_a^t f(s, y^{[k]}(s), u^{[k]}(s))\, ds \\ \gamma^{[k+1]} &= \frac{1}{b-a} \left(\beta - \alpha - \int_a^b (b-s) f(s, y^{[k]}(s), u^{[k]}(s))\, ds \right). \end{aligned} \tag{7b}$$

THEOREM

Theorem

Let $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R} : (t,y,u) \mapsto f(t,y,u)$ *be Lipschitz in* $\mathbf{y} = (y,u)$ *with Lipschitz constant L with respect to absolute value on* $\mathbb{R}$ *and the sum norm on* $\mathbb{R}^2$*. If* $0 < b - a < (1 + \frac{3}{2}L)^{-1}$ *then for any* $\alpha, \beta \in \mathbb{R}$ *the boundary value problem*

$$y'' = f(t,y,y'), \qquad y(a) = \alpha, \quad y(b) = \beta \tag{8}$$

has a unique solution.

EXAMPLES

In this example we treat a problem in which the function f on the right hand side fails to be Lipschitz yet Algorithm nevertheless performs well. Consider the boundary value problem

$$y'' = -e^{-2y}, \qquad y(0) = 0, \quad y(1.2) = \ln\cos 1.2 \approx -1.015\,123\,283, \tag{9}$$

for which the unique solution is $y(t) = \ln\cos t$, yielding $\gamma = 0$.
Introducing the dependent variable $u = y'$ to obtain the equivalent first order system $y' = u$, $u' = -e^{-2y}$ and the variable $v = e^{-2y}$ to replace the transcendental function with a polynomial we obtain the expanded system

$$\begin{aligned} y' &= u \\ u' &= -v \\ v' &= -2uv \end{aligned}$$

$$\begin{aligned} y^{[0]}(t) &\equiv 0 \\ u^{[0]}(t) &\equiv \frac{\ln\cos 1.2}{1.2} \\ v^{[0]}(t) &\equiv 1 \\ \gamma^{[0]} &= \frac{\ln\cos 1.2}{1.2} \end{aligned} \tag{10}$$

and

$$\begin{pmatrix} y^{[k+1]}(t) \\ u^{[k+1]}(t) \\ v^{[k+1]}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{1.2}(\beta - \alpha + \int_0^{1.2}(1.2-s)v^{[k]}(s)\,ds \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} u^{[k]}(s) \\ -v^{[k]}(s) \\ -2u^{[k]}(s)v^{[k]}(s) \end{pmatrix} ds, \tag{11}$$

where we have shifted the update of γ and incorporated it into the update of $u^{[k]}(t)$. The first eight iterates of γ are:

$$\gamma^{[1]} = -0.24594, \quad \gamma^{[2]} = \quad 0.16011, \quad \gamma^{[3]} = \quad 0.19297, \quad \gamma^{[4]} = 0.04165,$$

$$\gamma^{[5]} = -0.04272, \quad \gamma^{[6]} = -0.04012, \quad \gamma^{[7]} = -0.00923, \quad \gamma^{[8]} = 0.01030,$$

The maximum error in the approximation is $|y_{exact} - y^{[8]}|_{sup} \approx 0.0065$.

EXAMPLE

In this example the right hand side is not autonomous and does not satisfy a global Lipschitz condition. Consider the boundary value problem

$$y'' = \frac{1}{8}\left(32 + 2t^3 - yy'\right), \qquad y(1) = 17, \quad y(3) = \frac{43}{3} \tag{12}$$

The unique solution is $y(t) = t^3 + \frac{16}{t}$. We have

$$\begin{aligned} y^{[0]}(t) &\equiv 17 \\ u^{[0]}(t) &\equiv -\frac{4}{3} \\ \gamma^{[0]} &= -\frac{4}{3} \end{aligned} \tag{13a}$$

and

$$\begin{aligned} \begin{pmatrix} y^{[k+1]}(t) \\ u^{[k+1]}(t) \end{pmatrix} &= \begin{pmatrix} 17 \\ \gamma^{[k]} \end{pmatrix} + \int_1^t \begin{pmatrix} u^{[k]}(s) \\ 4 + \frac{1}{4}s^3 - y^{[k]}(s)u^{[k]}(s) \end{pmatrix} ds, \\ \gamma^{[k+1]} &= -\frac{4}{3} - \frac{1}{2}\int_1^3 (3-s)(4 + \frac{1}{4}s^3 - y^{[k]}(s)u^{[k]}(s))\, ds. \end{aligned} \tag{13b}$$

After $n = 12$ iterations, $\gamma^{[12]} = -8.443$, whereas the exact value is $\gamma = -\frac{76}{9} \approx -8.444$. As illustrated by the error plot, see the Figure in the next slide for the maximum error in the twelfth approximating function is $|y_{exact} - y^{[12]}|_{sup} \approx 0.0012$.

MIXED TYPE BOUNDARY CONDITIONS

The ideas developed at the beginning can also be applied to two-point boundary value problems of the form

$$y'' = f(t, y, y'), \qquad y'(a) = \gamma, \quad y(b) = \beta, \tag{14}$$

so that in equation (3) the constant γ is now known and $\alpha = y(a)$ is unknown. Thus in (5) we evaluate at $t = b$ but now solve for α instead of γ, obtaining in place of (6) the expression

$$\alpha = \beta - \gamma(b-a) - \int_a^b (b-s) f(s, y(s), y'(s))\, ds.$$

In the Picard iteration scheme we now successively update an approximation of α starting with some initial value α_0. The iterates are

$$\begin{aligned} y^{[0]}(t) &\equiv \alpha_0 \\ u^{[0]}(t) &\equiv \gamma \\ \alpha^{[0]} &= \alpha_0 \end{aligned} \tag{15a}$$

and

$$\begin{aligned} y^{[k+1]}(t) &= \alpha^{[k]} + \int_a^t u^{[k]}(s)\, ds \\ u^{[k+1]}(t) &= \gamma + \int_a^t f(s, y^{[k]}(s), u^{[k]}(s))\, ds \\ \alpha^{[k+1]} &= \beta - \gamma(b-a) - \int_a^b (b-s) f(s, y^{[k]}(s), u^{[k]}(s))\, ds. \end{aligned} \tag{15b}$$

Consider the boundary value problem

$$y'' = -y'e^{-y}, \qquad y'(0) = 1, \quad y(1) = \ln 2 \tag{16}$$

The exact solution is $y(t) = \ln(1+t)$, for which $y(0) = \ln 1 = 0$. Introducing the dependent variable $u = y'$ as always to obtain the equivalent first order system $y' = u$, $u' = -ue^{-y}$ and the variable $v = e^{-y}$ to replace the transcendental function with a polynomial we obtain the expanded system

$$\begin{aligned} y' &= u \\ u' &= -uv \\ v' &= -uv \end{aligned}$$

with initial conditions

$$y(0) = \alpha, \quad u(0) = \gamma, \quad v(0) = e^{-\alpha},$$

so that, making the initial choice $\alpha_0 = 1$

$$\begin{aligned} y^{[0]}(t) &\equiv \alpha_0 \\ u^{[0]}(t) &\equiv 1 \\ v^{[0]}(t) &\equiv e^{-\alpha_0} \end{aligned} \tag{17}$$

and

$$\begin{pmatrix} y^{[k+1]}(t) \\ u^{[k+1]}(t) \\ v^{[k+1]}(t) \end{pmatrix} = \begin{pmatrix} \ln 2 - 1 + \int_0^1 (1-s)u^{[k]}(s)v^{[k]}(s)\,ds \\ 1 \\ e^{-\alpha_0} \end{pmatrix} + \int_0^t \begin{pmatrix} u^{[k]}(s) \\ -u^{[k]}(s)v^{[k]}(s) \\ -u^{[k]}(s)v^{[k]}(s) \end{pmatrix} ds, \tag{18}$$

where we have shifted the update of α and incorporated it into the update of $y^{[k]}(t)$. We list the first eight values of $\alpha^{[k]}$ to show the rate of convergence to the exact value $\alpha = 0$.

$$\alpha^{[1]} = -0.00774, \quad \alpha^{[2]} = 0.11814, \quad \alpha^{[3]} = 0.07415, \quad \alpha^{[4]} = 0.08549,$$

$$\alpha^{[5]} = \quad 0.08288, \quad \alpha^{[6]} = 0.08339, \quad \alpha^{[7]} = 0.08330, \quad \alpha^{[8]} = 0.08331$$

Mixed Type Boundary Conditions: A Second Approach

In actual applications the Algorithm converges relatively slowly because of the update at every step of the estimate of the initial *value* $y(a)$ rather than of the initial *slope* $y'(a)$. A different approach that works better in practice with a problem of the type

$$y'' = f(t, y, y'), \qquad y'(a) = \gamma, \quad y(b) = \beta, \tag{19}$$

is to use the relation

$$y'(b) = y'(a) + \int_a^b f(s, y(s), y'(s))\, ds \tag{20}$$

between the derivatives at the endpoints to work from the right endpoint of the interval $[a,b]$, at which the value of the solution y is known but the derivative y' unknown. That is, assuming that (19) has a unique solution and letting the value of its derivative at $t = b$ be denoted δ, it is also the unique solution of the initial value problem

$$y'' = f(t, y, y'), \qquad y(b) = \beta, \quad y'(b) = \delta. \tag{21}$$

We introduce the new dependent variable $u = y'$ to obtain the equivalent system

$$y' = u$$
$$u' = f(t, y, u)$$

with initial conditions $y(b) = \beta$ and $y'(b) = \delta$ and apply Picard iteration based at $t = b$, using (20) with $y'(a) = \gamma$ to update the approximation of $y'(b)$ are each step. Choosing a convenient initial estimate δ_0 of δ, the successive approximations of the solution of (21), hence of (19), are given by

$$\begin{aligned} y^{[0]}(t) &\equiv \beta \\ u^{[0]}(t) &\equiv \delta_0 \\ \delta^{[0]} &= \delta_0 \end{aligned} \tag{22a}$$

and

$$\begin{aligned} y^{[k+1]}(t) &= \beta + \int_b^t u^{[k]}(s)\,ds \\ u^{[k+1]}(t) &= \delta^{[k]} + \int_b^t f(s, y^{[k]}(s), u^{[k]}(s))\,ds \\ \delta^{[k+1]} &= \gamma + \int_a^b f(s, y^{[k]}(s), u^{[k]}(s))\,ds. \end{aligned} \tag{22b}$$

EXAMPLE

Reconsider the boundary value problem

$$y'' = -y'e^{-y}, \qquad y'(0) = 1, \quad y(1) = \ln 2 \tag{23}$$

with exact solution $y(t) = \ln(1+t)$. The relation (20) is

$$y'(1) = 1 - \int_0^1 y'(s)e^{-y(s)}\,ds. \tag{24}$$

We set $u = y'$ and $v = e^{-y}$ to obtain the expanded system

$$\begin{aligned} y' &= u \\ u' &= -uv \\ v' &= -uv \end{aligned}$$

Since, our standard way is to have initial condition at $t = 0$. Therefore we use change of variable $\tau = 1 - t$ and Example becomes

$$y'' = y'e^{-y}, \qquad y(0) = ln2, \quad y'(1) = -1 \tag{25}$$

with exact solution $y(t) = \ln(\tau - 2)$. The relation (20) is

$$y'(1) = -1 + \int_0^1 y'(s)e^{-y(s)}\,ds. \tag{26}$$

We set $u = y'$ and $v = e^{-y}$ to obtain the expanded system

$$\begin{aligned} y' &= u \\ u' &= uv \\ v' &= -uv \end{aligned}$$

$$\begin{aligned} y^{[0]}(t) &\equiv \ln 2 \\ u^{[0]}(t) &\equiv -1 \\ v^{[0]}(t) &\equiv \tfrac{1}{2} \\ \delta^{[0]} &\equiv -11 \end{aligned} \tag{27a}$$

and

$$\begin{pmatrix} y^{[k+1]}(t) \\ u^{[k+1]}(t) \\ v^{[k+1]}(t) \end{pmatrix} = \begin{pmatrix} \ln 2 \\ \delta^{[k]} \\ \frac{1}{2} \end{pmatrix} + \int_0^t \begin{pmatrix} u^{[k]}(s) \\ u^{[k]}(s)v^{[k]}(s) \\ -u^{[k]}(s)v^{[k]}(s) \end{pmatrix} ds, \qquad \delta^{[k+1]} = -1 + \int_0^1 u^{[k]}(s)v^{[k]}(s)\,ds. \tag{27b}$$

The exact value of $\delta = y'(1)$ is $\frac{1}{2}$. The first eight values of $\delta^{[k]}$ are:

$$\delta^{[0]} = -0.50000, \quad \delta^{[1]} = 0.41667, \quad \delta^{[2]} = 0.50074, \quad \delta^{[3]} = 0.51592,$$

$$\delta^{[4]} = 0.49987, \quad \delta^{[5]} = 0.49690, \quad \delta^{[6]} = 0.50003, \quad \delta^{[7]} = 0.50060, \quad \delta^{[8]} = 0.50000$$

LONG INTERVALS

Consider the following Boundary Value Ordinary Differential Equation,

$$\begin{aligned} w''(t) = f(t, w(t), w'(t)), \qquad a \leq t \leq b \\ w(a) = \alpha, \qquad w(b) = \beta \end{aligned} \tag{28}$$

Here we introduce a new algorithm for the case if the original algorithm failed to converge on a long interval. We split the long interval into subintervals and show the new algorithm gives convergence to the solution. For simplicity we divided interval $[a, b]$ into three subintervals $[a, t_1]$, $[t_1, t_2]$ and $[t_2, b]$. Where $t_0 = a < t_1 < t_2 < t_3 = b$. Where $h = t_i - t_{i-1}$ for $1 \leq i \leq 3$

We would like to show that the boundary value problem (28) on the long interval has a unique solution if the following Initial value problems can simultaneously generate recursively a convergence sequence of $y(t)$ restricted to each subintervals.

$$\begin{aligned} w_1''(t) &= f(t, w_1(t), w_1'(t)), \qquad a \le t \le t_1 \\ w_1(a) &= \alpha, \qquad w_1'(a) = \gamma_1 \end{aligned} \tag{29}$$

$$\begin{aligned} w_2''(t) &= f(t, w_2(t), w_2'(t)), \qquad t_1 \le t \le t_2 \\ w_2(t_1) &= \beta_1, \qquad w_2'(t_1) = \gamma_2 \end{aligned} \tag{30}$$

and

$$\begin{aligned} w_3''(t) &= f(t, w_3(t), w_3'(t)), \qquad t_2 \le t \le b \\ w_3(t_2) &= \beta_2, \qquad w_3'(t_2) = \gamma_3 \end{aligned} \tag{31}$$

We convert interval $[t_{i-1}, t_i]$ to $[0, h]$ by $\tau = t - t_{i-1}$. This transformation can help us to simplify the proof. Hence (29) becomes

$$\begin{aligned} y_1''(\tau) &= f_1(\tau, y_1(\tau), y_1'(\tau)), \qquad 0 \le \tau \le h \\ y_1(0) &= \alpha, \qquad y_1'(0) = \gamma_1 \end{aligned} \tag{32}$$

(30) becomes

$$\begin{aligned} y_2''(\tau) &= f_2(\tau, y_2(\tau), y_2'(\tau)), \qquad 0 \le \tau \le h \\ y_2(0) &= \beta_1, \qquad y_2'(0) = \gamma_2 \end{aligned} \tag{33}$$

and (31) becomes

$$\begin{aligned} y_3''(\tau) &= f_3(\tau, y_3(\tau), y_3'(\tau)), \qquad 0 \le \tau \le h \\ y_3(0) &= \beta_2, \qquad y_3'(0) = \gamma_3 \end{aligned} \tag{34}$$

The initial conditions for each of the equations (32), (33) and (34) will be obtained so that the following conditions are satisfied

$$\begin{array}{lll} y_1(h;\beta_1,\gamma_1) & = y_2(0;\beta_1;\gamma_2) & \text{Continuity Condition} \\ y_2(h;\beta_2,\gamma_2) & = y_3(0;\beta_2;\gamma_3) & \text{Continuity Condition} \\ y_1'(h;\beta_1,\gamma_1) & = y_2'(0;\beta_1;\gamma_2) & \text{Smoothness Condition} \\ y_2'(h;\beta_2,\gamma_2) & = y_3'(0;\beta_2;\gamma_3) & \text{Smoothness Condition} \\ y_3(h;\beta_2;\gamma_3) & = \beta & \end{array} \tag{35}$$

These conditions will be satisfied automatically inside of the algorithm.
The recursion initialization becomes

$$\begin{aligned} y_1^{[0]}(\tau) &\equiv \alpha \\ u_1^{[0]}(\tau) &\equiv \frac{\beta-\alpha}{b-a} \\ \gamma_1^{[0]} &\equiv u_1^{[0]} \\ \beta_1^{[0]} &\equiv u_1^{[0]}h+\alpha \\ u_2^{[0]}(\tau) &\equiv \gamma_1^{[0]} \\ u_3^{[0]}(\tau) &\equiv \gamma_1^{[0]} \end{aligned} \tag{36}$$

$$\begin{pmatrix} y_1^{[k+1]}(\tau) \\ u_1^{[k+1]}(\tau) \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma_1^{[k]} \end{pmatrix} + \int_0^\tau \begin{pmatrix} u_1^{[k]}(s) \\ f_1(s, y_1^{[k+1]}(s), u_1^{[k]}(s)) \end{pmatrix} ds \tag{37a}$$

$$\phi_1^{[k]} \equiv y_1^{[k+1]}(h), \qquad \gamma_2^{[k]} \equiv u_1^{[k+1]}(h) \tag{37b}$$

$$\begin{pmatrix} y_2^{[k+1]}(\tau) \\ u_2^{[k+1]}(\tau) \end{pmatrix} = \begin{pmatrix} \phi_1^{[k]} \\ \gamma_2^{[k]} \end{pmatrix} + \int_0^\tau \begin{pmatrix} u_2^{[k]}(s) \\ f_2(s, y_2^{[k+1]}(s), u_2^{[k]}(s)) \end{pmatrix} ds \tag{37c}$$

$$\phi_2^{[k]} \equiv y_2^{[k+1]}(h), \qquad \gamma_3^{[k]} \equiv u_2^{[k+1]}(h) \tag{37d}$$

$$\begin{pmatrix} y_3^{[k+1]}(\tau) \\ u_3^{[k+1]}(\tau) \end{pmatrix} = \begin{pmatrix} \phi_2^{[k]} \\ \gamma_3^{[k]} \end{pmatrix} + \int_0^\tau \begin{pmatrix} u_3^{[k]}(s) \\ f_3(s, y_3^{[k+1]}(s), u_3^{[k]}(s)) \end{pmatrix} ds \tag{37e}$$

$$\begin{aligned} \beta_2^{[k]} &= \beta - \gamma_3^{[k]} h - \int_0^h (h-s) f_3(s, y_3^{[k+1]}(s), u_3^{[k+1]}(s))\, ds \\ \beta_1^{[k+1]} &= \beta_2^{[k]} - \gamma_2^{[k]} h - \int_0^h (h-s) f_2(s, y_2^{[k+1]}(s), u_2^{[k+1]}(s))\, ds \\ \gamma_1^{[k+1]} &= \frac{1}{h}\left[\beta_1 - \alpha - \int_0^h (h-s) f_1(s, y_1^{[k+1]}(s), u_1^{[k+1]}(s))\, ds\right]. \end{aligned} \tag{37f}$$

Example

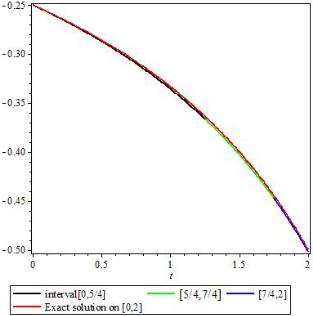
Consider the boundary value problem, we

$$y'' = 2y^3, \qquad y(0) = -\frac{1}{4}, \qquad y(2) = \beta \tag{38}$$

Our goal is to divide a long intervals into n subintervals, with relatively small length, that satisfies our original Theorem. We divided the interval into three subintervals $[0, \frac{5}{4}] \cup [\frac{5}{4}, \frac{7}{4}] \cup [\frac{7}{4}, 2]$. For $n = 18$.
The exact value of $\gamma = -0.0625$. The first 10 values of $\gamma^{[k]}$ are:

$$\gamma^{[1]} = -0.12500, \quad \gamma^{[2]} = -0.10156, \quad \gamma^{[3]} = -0.06154, \quad \gamma^{[4]} = -0.04716, \quad \gamma^{[5]} = -0.06917,$$

$$\gamma^{[6]} = -0.06903, \quad \gamma^{[7]} = -0.05852, \quad \gamma^{[8]} = -0.05948, \quad \gamma^{[9]} = -0.06515, \quad \gamma^{[10]} = -0.06366,$$


-0.25
-0.30
-0.35
-0.40
-0.45
-0.50
0
0.5
1
1.5
2
t
interval[0,5/4]
[5/4,7/4]
[7/4,2]
Exact solution on [0,2]

VOLTERRA INTEGRAL

A Volterra equation of the second kind is an equation of the form

$$y(t) = \varphi(t) + \int_0^t K(t,s)f(s,y(s))\,ds \tag{39}$$

where φ, K, and f are known functions of suitable regularity and y is an unknown function. Such equations lend themselves to solution by successive approximation using Picard iteration, although in the case that the known functions are not polynomials the process can break down when quadratures that cannot be performed in closed form arise. If the “kernel“ K is independent of the first variable t then the integral equation is equivalent to an initial value problem. Indeed, in precisely the reverse of the process.

We have that

$$y(t) = \varphi(t) + \int_0^t k(s)f(s, y(s))\,ds$$

is equivalent to

$$y'(t) = \varphi'(t) + k(t)f(t, y(t)), \quad y(0) = \varphi(0).$$

In this chapter we provide a method for introducing auxiliary variables in (39), in the case that K factors as $K(t,s) = j(t)k(s)$, in such a way that it embeds in a vector-valued polynomial Volterra equation, thus extending the Parker-Sochacki method to this setting and thereby obtaining a computationally efficient method for closely approximating solutions of (39).

EXAMPLE (VIE)

Linear

Consider the following second kind linear volterra integral.

$$y(t) = \exp(t)\sin(t) + \int_0^t \frac{2+\cos(t)}{2+\cos(s)} y(s)ds \tag{40}$$

$$\begin{cases} \phi(t) = \exp(t)\sin(t) \\ \\ k(t,s) = \frac{2+\cos(t)}{2+\cos(s)} \\ f(t,y(t)) = y(t) \end{cases} \tag{41}$$

The exact solution is

$$y(t) = \exp(t)\sin(t) + \exp(t)\Big(2+\cos(t)\Big)\Big(\ln(3) - \ln\big(2+\cos(t)\big)\Big) \tag{42}$$

EXAMPLE

$$\begin{cases} y(t) &= \exp(t)\sin(t) + (2+\cos(t))\int_0^t \frac{1}{2+\cos(s)} y(s)ds \\ y(0) &= \exp(0)\sin(0) = 0 \end{cases} \tag{43}$$

We define the following variables

$$\begin{aligned} v_1 &= \exp(t), \qquad v_1(0) = 1 \\ v_2 &= \cos(t), \qquad v_2(0) = 1 \\ v_3 &= \sin(t), \qquad v_3(0) = 0 \\ v_4 &= 2 + v_2, \qquad v_4(0) = 3 \\ v_5 &= \frac{1}{v_4}, \qquad v_5(0) = \frac{1}{3} \end{aligned} \tag{44}$$

- $v_1' = v_1$
- $v_2' = -v_3$
- $v_3' = v_2$
- $v_4' = v_2' = -v_3$
- $v_5' = \frac{-v_4'}{v_4^2} = v_3 v_5^2$

Example

- $v_1(t) = v_1(0) + \int_0^t v_1(s)ds$
- $v_2(t) = v_2(0) - \int_0^t v_3(s)ds$
- $v_3(t) = v_3(0) + \int_0^t v_2(s)ds$
- $v_4(t) = v_4(0) - \int_0^t v_3(s)ds$
- $v_5(t) = v_5(0) + \int_0^t v_3(s)v_5^2 ds$

- $y^{[1]} = 0 = \alpha$
- $v_1^{[1]} = \exp(0) = 1$
- $v_2^{[1]} = \cos(0) = 1$
- $v_3^{[1]} = \sin(0) = 0$
- $v_4^{[1]} = 2 + v_2^{[1]} = 3$
- $v_5^{[1]} = \frac{1}{3}$

Example VIE

Thus the iterates are

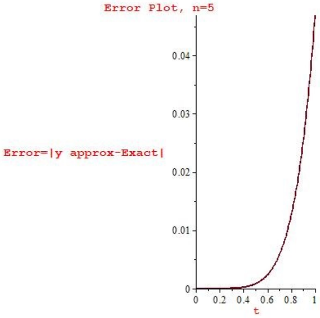
- $y^{[k+1]} = q_1^{[k]} v_3^{[k]} + v_4^{[k]} \int_0^t v_5^{[k]} y^{[k]} ds$
- $v_1^{[k+1]} = 1 + \int_0^t v_1^{[k]} ds$
- $v_2^{[k+1]} = 1 - \int_0^t v_3^{[k]} ds$
- $v_3^{[k+1]} = \int_0^t v_2^{[k]} ds$
- $v_4^{[k+1]} = 3 - \int_0^t v_3^{[k]} ds$
- $v_5^{[k+1]} = \frac{1}{3} + \int_0^t v_3^{[k]} (v_5^{[k]})^2 ds$

The approximation of solution for the first 4 iterations

$$
\begin{aligned}
y^{[2]} &= 0 \\
y^{[3]} &= t^2 + t \\
y^{[4]} &= -\frac{1}{180} t^7 - \frac{1}{144} t^6 - 1/45\, t^5 - 1/24\, t^4 + 5/6\, t^3 + 3/2\, t^2 + t \\
y^{[5]} &= \frac{1}{9797760} t^{16} + \frac{1}{7278336} t^{15} + \frac{59}{29393280} t^{14} + \frac{41}{13343616} t^{13} \\
&\quad - \frac{1}{87480} t^{12} - \frac{1}{42768} t^{11} - \frac{1}{2880} t^{10} - \frac{1}{1512} t^9 - \frac{1}{480} t^8 - \frac{1}{336} t^7 \\
&\quad - \frac{49}{1080} t^6 - 1/8\, t^5 + 1/6\, t^4 + 5/6\, t^3 + 3/2\, t^2 + t
\end{aligned} \tag{45}
$$

EXAMPLE VIE

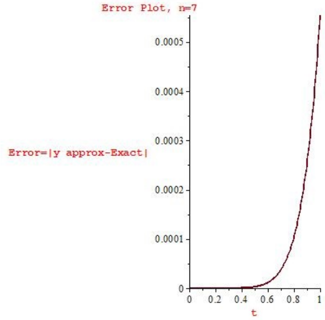
For $n = 5$



$y_{approx} = y^{[n+1]}, \qquad Exact = y(t)$

EXAMPLE VIE

By increasing n tto $n = 7$



$y_{approx} = y^{[n+1]}$, $Exact = y(t)$

Example VIE

Non-Linear

Biazar on his paper [11] has introduced the following second kind non-linear volterra integral.

$$\begin{cases} \phi(t) = 1/2\sin(2t) \\ k(t,s) = \cos(s-t) \\ f(t,y) = y^2 \end{cases} \tag{46}$$

The exact solution is

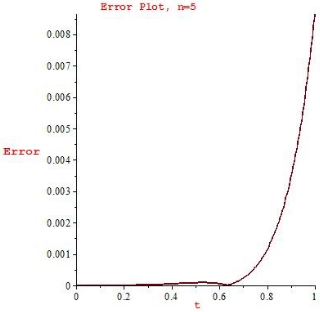
$$y(t) = \sin(t) \tag{47}$$

To solve this non-linear Volterra integral we set up the integral

$$\begin{cases} y(t) = \frac{1}{2}\sin(2t) + \frac{3}{2}\int_0^t \cos(s-t)y^2(s)ds \\ y(0) = \frac{1}{2}\sin(2(0)) = 0 \end{cases} \tag{48}$$

EXAMPLE VIE

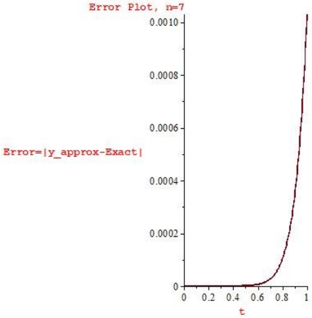
For $n = 5$



$$Error = |y^{[n+1]} - y(t)|$$

EXAMPLE VIE

By increasing n to $n = 7$ we have



$y_{approx} = y^{[n+1]}$, $\quad Exact = y(t)$

Thank you

Thank you

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