Poincaré-Betti Series of Monomial Quotient Rings

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Free Resolutions over Polynomial Rings

Definition

Let $R = \Bbbk[x_1, \ldots, x_n]$ ring and $I \subseteq R$ an ideal. Then $\mathcal{F}$, the minimal graded free resolution of R/I is a chain complex of the form

$$\mathcal{F}: \quad R/I \xleftarrow{\varphi_0} R \xleftarrow{\varphi_1} \bigoplus_{j \geq 0} R(-j)^{\beta_{1,j}} \xleftarrow{\varphi_2} \bigoplus_{j \geq 0} R(-j)^{\beta_{2,j}} \longleftarrow \cdots,$$

with $\varphi_i : F_i \to F_{i-1}$ degree zero maps with entries in the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$.

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$$\beta^R_{0,0}(M) = 1$$

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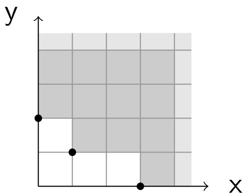

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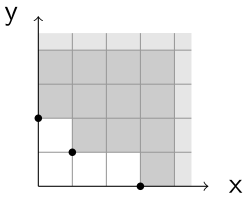

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Instead of resolving R/M over R, we now consider the problem of resolving $\mathbb{k} = S/\mathfrak{m}$ (where the maximal ideal $\mathfrak{m} = (x, y)$) over $S = R/M$ where $M = (x^2, xy)$.

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The resolution of $\Bbbk$ is infinite (by Auslander-Buchsbaum-Serre.)

Poincaré-Betti Series of Modules

Definition (Poincare-Betti Series)

The graded Poincaré-Betti series of $\mathbb{k}$ over S, denoted by $P_S(z)$, is the formal power series

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Example for $R/M = \Bbbk[x,y]/(x^2, xy)$

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For $R = \Bbbk[x,y]$ and $M = (x^2, xy)$, we have the Poincaré-Betti series of $\Bbbk$ over $S = R/M$ is:

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Resolutions of $\Bbbk$ over S

Theorem (Resolutions of $\Bbbk$ over Quotient Rings S)

Let M be an r-generated monomial ideal with $r > 2$, or $r = 2$ and M not generated by pure powers of x and y, $(\deg(m_i) \geq 2\ \forall i)$.

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with $\partial_{i+1} = \partial_3^{v_i} \oplus \partial_2^{w_i + r\cdot u_i} \oplus \partial_1^{(r-1)\cdot u_i}$.

Recursion Formulas for Betti Sequences

Corollary (Poincare-Betti Polynomials)

Let M be an r-generated monomial ideal of $\Bbbk[x, y]$, M not generated by pure powers of x and y. The total Betti numbers of the resolution of $\Bbbk$ over $S = R/M$ are given by

$$\beta_i^S(\Bbbk) = \begin{cases} 1 & \text{if } i = 0, \\ 2 & \text{if } i = 1, \\ \beta_{i-1}^S(\Bbbk) + (r-1)\beta_{i-2}^S(\Bbbk) & \text{if } i \geq 2. \end{cases}$$

Recursion Formulas and Poincaré-Betti Series

So the graded Poincaré-Betti series of S is

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$$P_S(z) = \frac{(1+z)^2}{b_S(z)}, \quad b_S(z) = 1 - rz^2 + (1-r)z^3$$

Previous and Future Work

- Not all Poincaré-Betti series of rings R/I for ideals $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$ are rational. [Roos-Sturmfels '98, Fröberg-Roos '99]

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- Alternate combinatorial formula for in terms of the homologies of a related intersection lattice of a subspace arrangement lattice. [Berglund-Blasiak-Hersh '07]

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MAA MD-VA-DC Section Meeting: Fall 2013

MAA Section Meeting, Fall 2013

Thanks for listening!