On a Formula of Liouville Type for the Quadratic Form $x^2 + 2y^2 + 2z^2 + 4w^2$

Cherng-tiao Perng

Norfolk State University

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A Brief History

- In 1772, Euler gave a simplified proof of Lagrange's Theorem of Four Squares.
- In 1834, Jacobi gave a proof of a formula for the number of representations of a positive integer as a sum of four integer squares.
- In 1843, Hamilton discovered quaternions.
- In 1886, Lipschitz gave a quaternionic proof of Jacobi's formula.
- In 1896, Hurwitz gave another quaternionic proof of Jacobi's formula.
- In 2004, J. Deutsch gave a quaternionic proof for the representation formula associated with the quadratic form $x^2 + y^2 + 2z^2 + 2w^2$ based on analogues of Hurwitz quaternions.

Some of My Recent Work

• Inspired by Lipschitz's work, the author gave in 2011 a variant proof of the Jacobi's formula for the number of representations of a positive integer as a sum of four integer squares.

• In 2012, the author gave a quaternionic proof for the number of representations associated with the quadratic forms $x^2 + y^2 + 2z^2 + 2w^2$ and $x^2 + y^2 + 3z^2 + 3w^2$ based on Lipschitz type quaternions.

• In 2012, the author gave a quaternionic proof for the number of representations associated with the quadratic form $x^2 + 2y^2 + 2z^2 + 4w^2$.

The Representation Formula for the Quadratic Form $x^2 + 2y^2 + 2z^2 + 4w^2$

This was first proposed by Liouville (1862), so we call it a formula of Liouville type.

Theorem. Let $n = 2^\alpha N$. Then the number S of representations of n in terms of the quadratic form $x^2 + 2y^2 + 2z^2 + 4w^2$ is given by

$$S = \begin{cases} 2\sigma(N) & \text{if } \alpha = 0, \\ 4\sigma(N) & \text{if } \alpha = 1, \\ 8\sigma(N) & \text{if } \alpha = 2, \\ 24\sigma(N) & \text{if } \alpha \geq 3, \end{cases}$$

where σ is the sum of divisors function.

Lipschitz Quaternions

For the purpose of this talk, we briefly mention the concept of the set of Lipschitz quaternions. It is a ring generated by the symbols i, j and k, subject to the multiplication rules $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. More precisely, any Lipschitz quaternion is of the form $a + bi + cj + dk$, where a, b, c, d are integers. It can be shown that the multiplication is associated but non-commutative.

For any Lipschitz quaternion $Q = a + bi + cj + dk$, we define its conjugate $\overline{Q}$ by $\overline{Q} = a - bi - cj - dk$. Furthermore, we define its norm by $\mathrm{Nm}(Q) = \overline{Q}Q = Q\overline{Q} = a^2 + b^2 + c^2 + d^2$. We will need also the notion of reduction modulo a prime p, in which case, the components of the quaternions would stay in the finite field $\mathbb{F}_p$.

Lipschitz Type Quaternions

For proving the Liouville type formula stated above, we need the following generalization of Lipschitz quaternions:

Definition. Let $\mathbb{L} = \{a + b\sqrt{2}i + c\sqrt{2}j + 2dk \mid a, b, c, d \in \mathbb{Z}\}$.

It is easy to show that $\mathbb{L}$ is closed under multiplication. We define also the conjugate and norm for $Q = a + b\sqrt{2}i + c\sqrt{2}j + 2dk \in \mathbb{L}$ as follows:

$$\overline{Q} = a - b\sqrt{2}i - c\sqrt{2}j - 2dk$$

and

$$\mathrm{Nm}(Q) = Q\overline{Q} = a^2 + 2b^2 + 2c^2 + 4d^2.$$

Further Definitions

We adopt a shorthand notation for $Q = a + b\sqrt{2}i + c\sqrt{2}j + 2dk$ by $[a, b, c, d]$.

Definition. A quaternion $[a, b, c, d] = Q \in \mathbb{L}$ is primitive if $\gcd(a, b, c, d) = 1$.

Definition. A quaternion $Q \in \mathbb{L}$ is a unit if $\mathrm{Nm}(Q) = 1$. It is easy to check that $\mathbb{L}$ has only two units, i.e. ± 1.

Definition. Two quaternions $Q_1, Q_2 \in \mathbb{L}$ are equivalent if there exists a unit $\epsilon \in \mathbb{L}$ such that $Q_2 = \epsilon Q_1$.

Definition. We say $Q \in \mathbb{L}$ is a prime quaternion if $\mathrm{Nm}(Q)$ is a rational prime.

Definition. We say that $Q \in \mathbb{L}$ is p-pure if $\mathrm{Nm}(Q) = p^r$.

Correspondence Theorem

Lemma. Let $p > 2$ be a prime. There are precisely $p + 1$ projective solutions for

$$x^2 + 2y^2 + 2z^2 = 0 \text{ over } \mathbb{F}_p.$$

We will lift the solutions to $\mathbb{Z}$ and represent them in the form of p-primitive quaternion $X = [x, y, z, 0]$ (i.e. $x + y\sqrt{2}i + z\sqrt{2}j$, $x, y, z \in \mathbb{Z}$, not all zero mod p).

Correspondence Theorem (Continued)

Let $p > 2$. Let S be the set of projective solutions of $x^2 + 2y^2 + 2z^2 = 0$ over $\mathbb{F}_p$ and T be the set of equivalence classes of prime quaternions of norm p.

Theorem. (Correspondence Theorem) Let $p > 2$. Then there exists a naturally defined bijection $\Psi : S \to T$. In particular, there are precisely $p + 1$ equivalence classes of prime quaternions of norm p.

Unique Factorization

Theorem. (a) Any primitive quaternion Q of norm $2^{s_0}p_1^{s_1}\cdots p_k^{s_k}$ can be factored uniquely under the standard model, namely

$$Q = \epsilon Q_0 Q_1 \cdots Q_k,$$

where ϵ is a unit, $Q_0 = 1$ (if $s_0 = 0$) or one of the representatives of the primitive quaternions of norm 2^{s_0}, and for $1 \leq i \leq k$, Q_i is a product of s_i's prime quaternions from the set of representatives of equivalence classes of prime quaternions of norm p_i.

Unique Factorization (Continued)

(b) Any non-primitive quaternion $Q' = mQ$ (with $m > 1$ and Q primitive of norm given as above) can be factored uniquely in the form

$$Q' = \epsilon(2^{t_0} Q_0)(p_1^{t_1} Q_1) \cdots (p_k^{t_k} Q_k)$$

under the model $2^{r_0} p_1^{r_1} \cdots p_k^{r_k}$ (still called standard), where $r_i = 2t_i + s_i, 0 \leq i \leq k$ and $m = 2^{t_0} p_1^{t_1} \cdots p_k^{t_k}$, and $Q_i, 0 \leq i \leq k$ is as described in (a).

Main Ideas of the Proof

- The formula $\mathrm{Nm}([x, y, z, w]) = x^2 + 2y^2 + 2z^2 + 4w^2$ indicates that the number of representations of n in terms of the quadratic form $x^2 + 2y^2 + 2z^2 + 4w^2$ equals the number of quaternions in $\mathbb{L}$ of norm n.

- This motivates the study of factorization in $\mathbb{L}$.

- The factorization of 2-pure $Q \in \mathbb{L}$ into factors of prime quaternions may not always work, hence we consider only 2-pure primitive quaternions as part of the building blocks in the factorization.

Main Ideas of the Proof (Continued)

• The proof of the representation formula for the quadratic form $x^2 + 2y^2 + 2z^2 + 4w^2$ is based on the Unique Factorization, where we build a factorization by a unit, a representative of 2-pure primitive quaternions, and the product of representatives of quaternions of odd prime norm.

• Since we know how to count the number of equivalence classes of 2-pure quaternions, and the number of equivalence classes of p-pure quaternions (for $p > 2$) based on the Correspondence Theorem, the representation formula follows easily.

For details, please refer to my paper "On a Formula of Liouville Type for the Quadratic Form $x^2 + 2y^2 + 2z^2 + 4w^2$", International Mathematical Forum, Vol. 8, 2013, no. 33, 1605 - 1614.

Example

We give an example when $n = 24 = 2^3 \cdot 3$. By brute-force search for the number of vectors (x, y, z, w) with $x^2 + 2y^2 + 2z^2 + 4w^2 = 24$, we get that $S = 96$. See the following SAGE code for computation:

```
S = 0
 for i in range(-4,5):
   for j in range(-3,4):
     for k in range(-3,4):
       for l in range(-2,3):
         if norm((i,j,k,l))==24:
           S = S +1
S = 96
```

Example (Continued)

Note that $n = 2^3 \cdot 3$.

• There are precisely $4 = 3 + 1$ equivalence classes of the Lipschitz type quaternions of norm 3: $1 + \sqrt{2}i$, $1 - \sqrt{2}i$, $1 + \sqrt{2}j$ and $1 - \sqrt{2}j$.

• There are precisely 12 equivalence classes of the Lipschitz type quaternions of norm 8: 2 classes of the form $2Q$, where Q is primitive of $\mathrm{Nm}(Q) = 2$, and 10 classes of primitive Q such that $\mathrm{Nm}(Q) = 8$. More precisely, these are represented by $2\sqrt{2}i, 2\sqrt{2}j, 2 \pm 2k, \sqrt{2}i \pm \sqrt{2}j \pm 2k$, and $2 \pm \sqrt{2}i \pm \sqrt{2}j$.

By our construction, $S = 2 \cdot 12 \cdot 4 = 24\sigma(3) = 96$.