Mathematical Association of America
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An Arithmetic View of a Classical Calculus Problem

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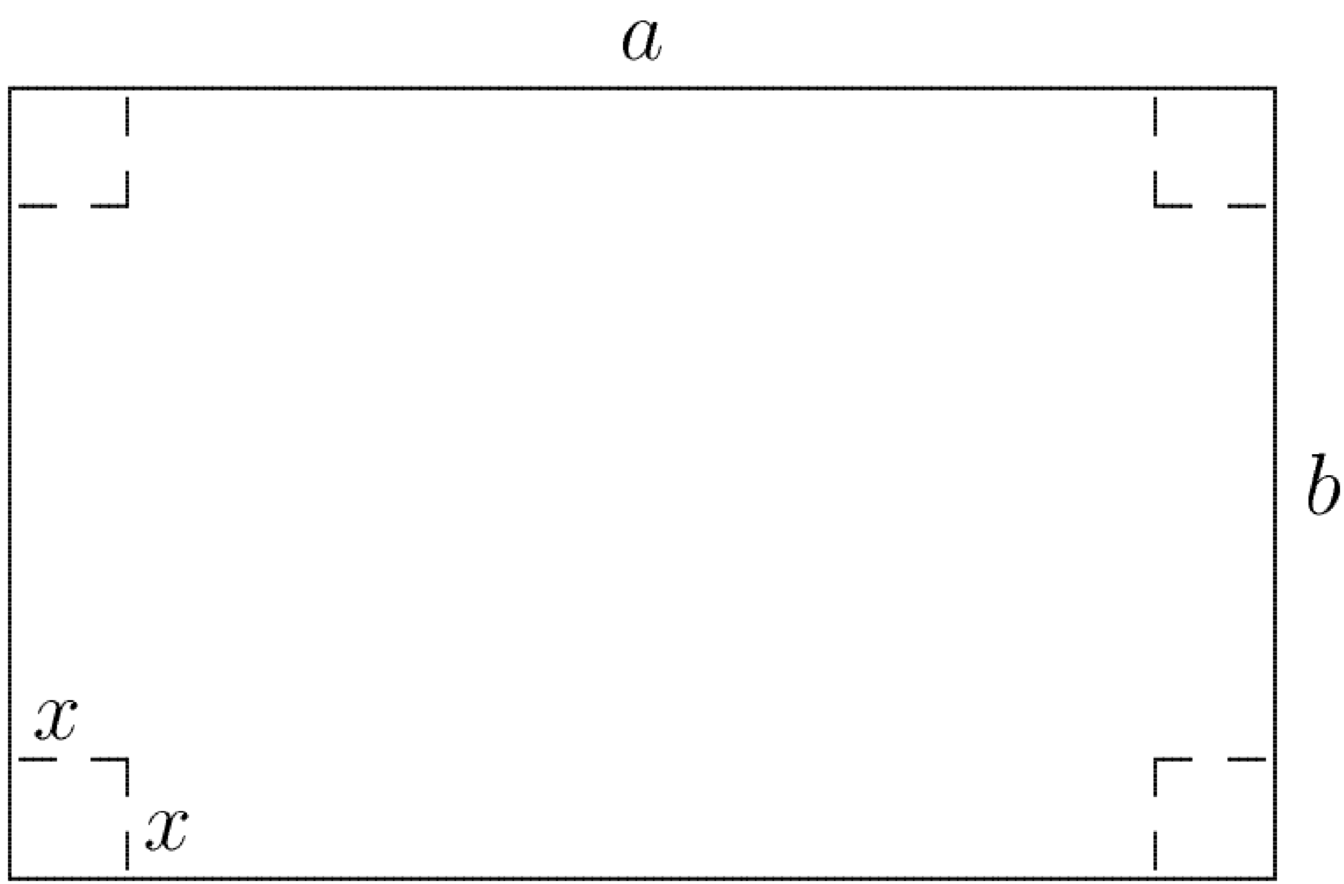
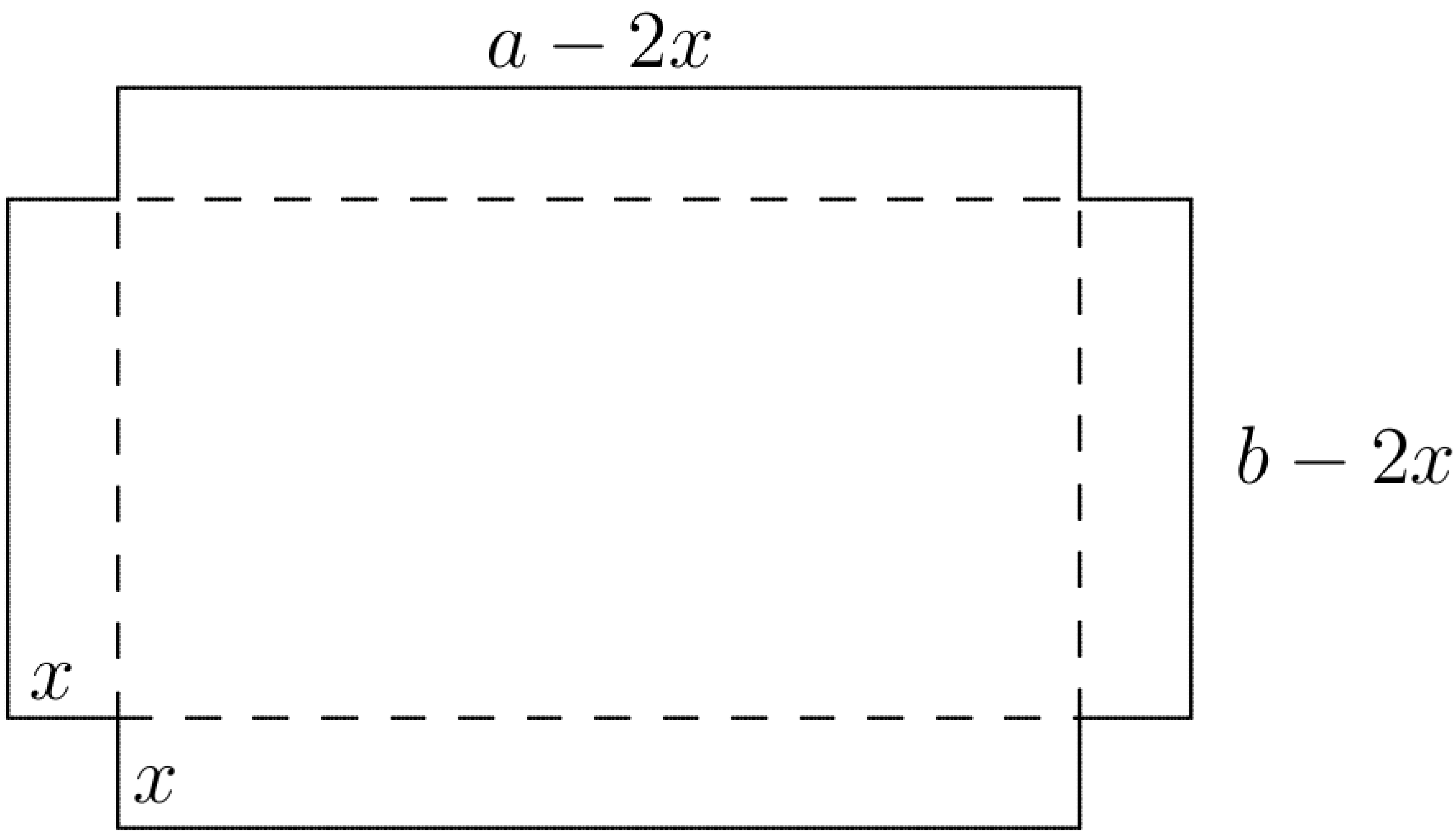
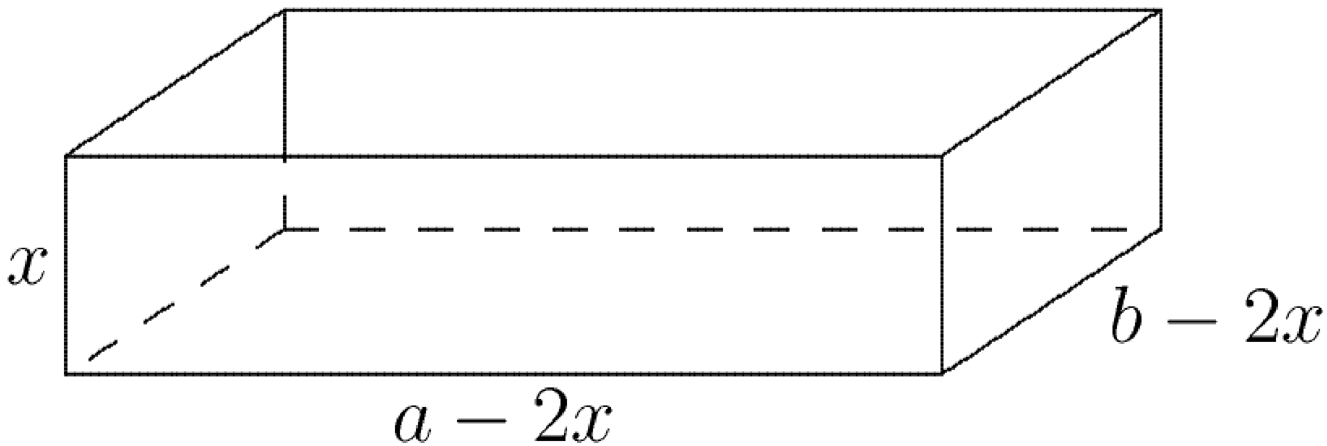
Calculus problem



Squares cut from each corner



Flaps folded to form open box



$$V = x(a - 2x)(b - 2x) = 4x^3 - 2(a + b)x^2 + abx$$

Calculus solution

$$V'(x) = 0 \text{ if and only if } x = \frac{a+b\pm c}{6},$$
$$\text{where } c = \sqrt{a^2 - ab + b^2}.$$

If $a \geq b > 0$, then $a \geq c \geq b$, so that

$$\frac{a+b-c}{6} \text{ is the only critical point in } \left(0, \frac{b}{2}\right).$$

Arithmetic question

For what integers a and b is $x = \frac{a+b-c}{6}$
a rational number?

This is true if and only if
$c = \sqrt{a^2 - ab + b^2}$ is a rational integer.

Find all integer solutions of $a^2 - ab + b^2 = c^2$
with $a > b > 0$, and $\gcd(a, b, c) = 1$.

Main result

Let $0 < r < s$ be integers with $\gcd(r, s) = 1$ and $r \not\equiv s \pmod 3$, and let $t = r + s$. Then

$$(1) \qquad (a, b, c) = (t^2 - r^2, t^2 - s^2, t^2 - rs)$$

and

$$(2) \qquad (a, b, c) = (t^2 - r^2, s^2 - r^2, t^2 - rs)$$

are primitive solutions of $a^2 - ab + b^2 = c^2$

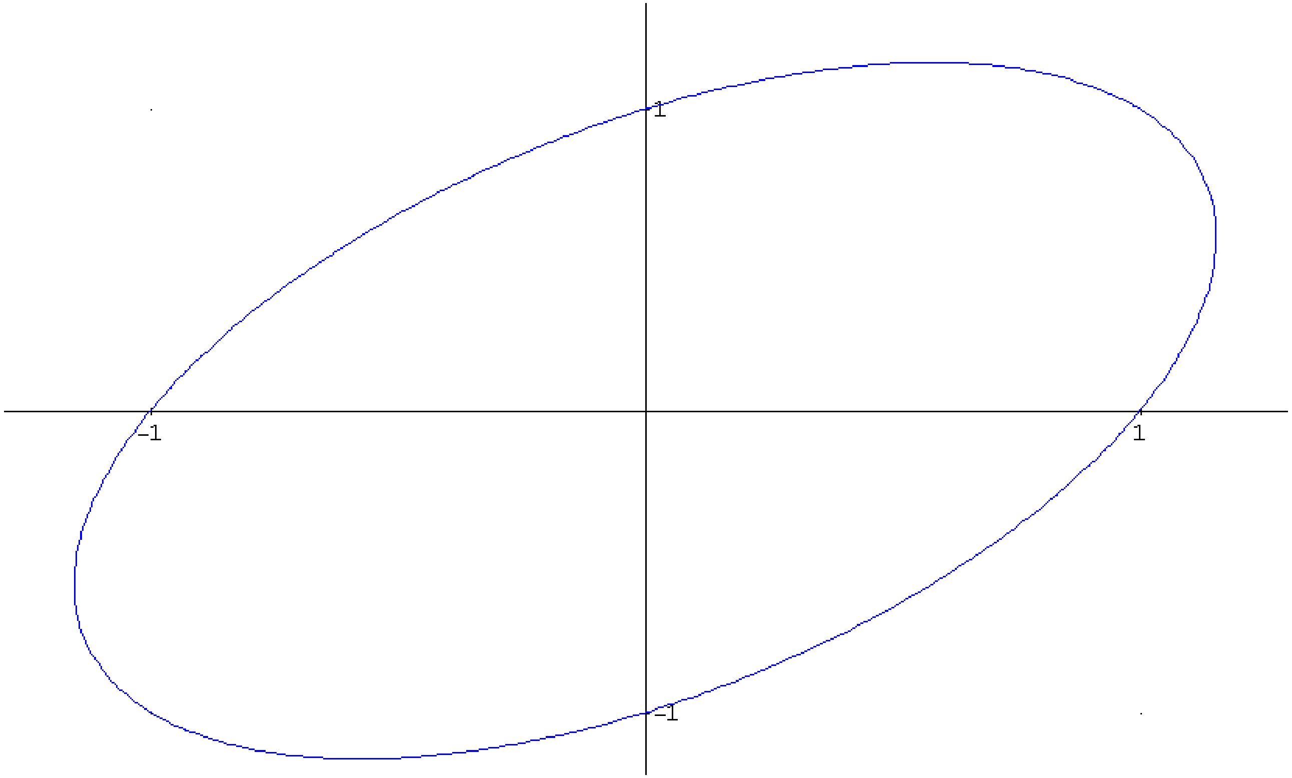
All primitive solutions are obtained in this way.

Arithmetic question rephrased

Find all rational points $(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right)$

on the ellipse $x^2 - xy + y^2 = 1$ with $x > 1$.

Graph of the ellipse



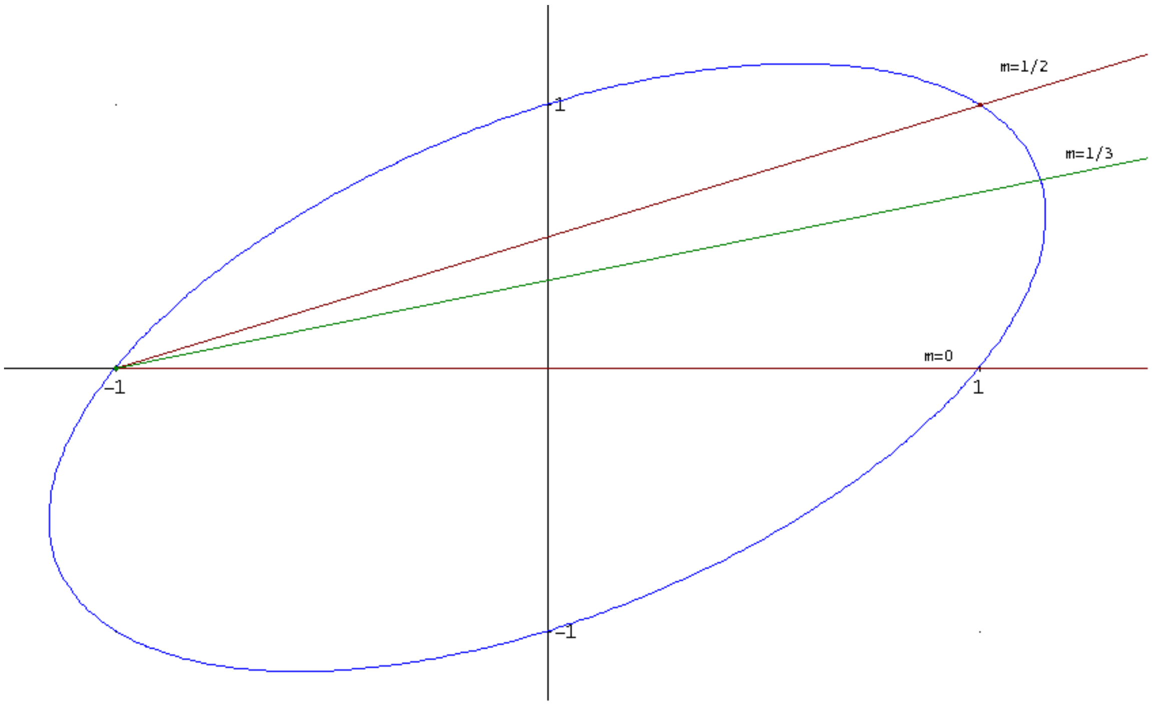
$$x^2 - xy + y^2 = 1$$

Geometric approach

The line joining a rational point (x, y) and $(-1, 0)$ has rational slope.

If m is a rational number, the line through $(-1, 0)$ with slope m meets the ellipse at a rational point.

Example



$$x^2 - xy + y^2 = 1 \text{ and } y = \frac{1}{3}(x+1)$$
$$\text{intersect at } \left(\frac{8}{7}, \frac{5}{7}\right)$$

Points of intersection

Intersections of $y = m(x+1)$ and $x^2 - xy + y^2 = 1$:

$$x^2 - x \cdot m(x+1) + m^2(x+1)^2 = 1$$
$$(x^2 - 1) - mx(x+1) + m^2(x+1)^2 = 0$$
$$(m^2 - m + 1)x^2 + (2m^2 - m)x + (m^2 - 1) = 0$$

$$(x+1)[(x-1) - mx + m^2(x+1)]$$
$$= (x+1)[(1 - m + m^2)x - (1 - m^2)] = 0$$

Rational points on the ellipse (x>1)

$$(x, y) = \left(\frac{1 - m^2}{1 - m + m^2}, \frac{2m - m^2}{1 - m + m^2} \right)$$

where m is a rational number with $0 < m < \frac{1}{2}$.

$$\left(y = m(x + 1) = \frac{2m - m^2}{1 - m + m^2}. \right)$$

Symmetry of rational points

If (x, y) is a rational point on $x^2 - xy + y^2 = 1$,
then so is $(x, x - y)$.

$$\text{So } (x, y) = \left(\frac{1 - m^2}{1 - m + m^2}, \frac{1 - 2m}{1 - m + m^2} \right)$$

is also a formula for all rational points
on $x^2 - xy + y^2 = 1$ with $x > 1$, when $0 < m < \frac{1}{2}$.

Integer solutions (a,b,c)

If $m = \dfrac{r}{t}$ for integers $t \geq 2r \geq 0$ with $\gcd(r,t) = 1$, then

$$(a,b,c) = (t^2 - r^2, 2rt - r^2, r^2 - rt + t^2)$$

and

$$(a,b,c) = (t^2 - r^2, t^2 - 2rt, r^2 - rt + t^2)$$

are integer solutions of $a^2 - ab + b^2 = c^2$.

Here $\gcd(a,b,c) = 1$ if $t \not\equiv 2r \pmod 3$, while

$\gcd(a,b,c) = 3$ if $t \equiv 2r \pmod 3$.

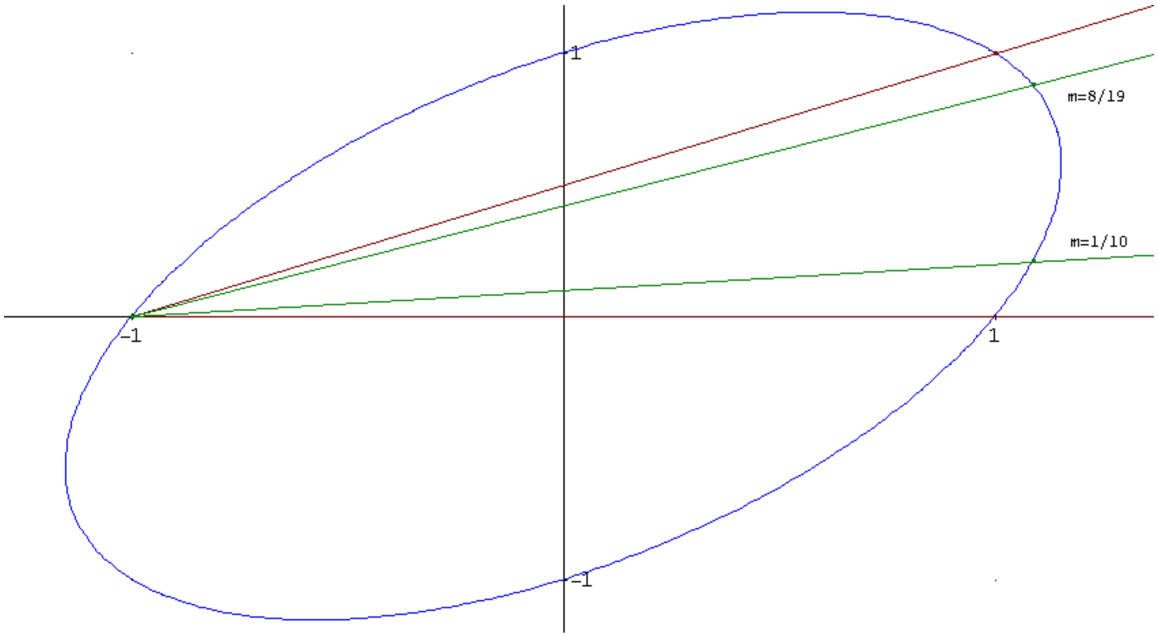
Relation to slopes

If $m = \dfrac{r}{t}$ corresponds to the point (x, y),

then $m' = \dfrac{t - 2r}{2t - r} = \dfrac{r'}{t'}$ corresponds to $(x, x - y)$.

Exactly one of these slopes satisfies $r \not\equiv 2t \pmod 3$.

Example



$$\left(\frac{99}{91}, \frac{19}{91}\right) \text{ and } \left(\frac{99}{91}, \frac{80}{91}\right)$$

correspond to $m = \frac{1}{10}$ and $m = \frac{8}{19}$ respectively.

Change of notation

So we can restrict our attention to $r \not\equiv 2t \pmod 3$.

Finally, if $s = t - r$, so that $r \not\equiv s \pmod 3$, then

$$t^2 - s^2 = t^2 - (t^2 - 2rt + r^2) = 2rt - r^2,$$
$$s^2 - r^2 = t^2 - 2rt + r^2 - r^2 = t^2 - 2rt,$$
$$t^2 - rs = t^2 - r(t - r) = t^2 - rt + r^2.$$

Main result

Let $0 < r < s$ be integers with $\gcd(r, s) = 1$ and $r \not\equiv s \pmod 3$, and let $t = r + s$. Then

$$(1) \qquad (a, b_1, c) = (t^2 - r^2, t^2 - s^2, t^2 - rs)$$

and

$$(2) \qquad (a, b_2, c) = (t^2 - r^2, s^2 - r^2, t^2 - rs)$$

are primitive solutions of $a^2 - ab + b^2 = c^2$

All primitive solutions are obtained in this way.

Calculus solution implications

In case (1), $x_1 = \dfrac{a+b-c}{6} = \dfrac{rs}{2}$.

In case (2), $x_2 = \dfrac{a+b-c}{6} = \dfrac{(s-r)(s+2r)}{6}$.

If r or s is even, then $x_1 = n$ with $n \equiv 0, 1 \pmod{3}$.

If r and s are odd, then $x_1 = \frac{n}{2}$ with $n \equiv 3, 5 \pmod{6}$.

If s is even or $r \equiv s \pmod{2}$, then $x_2 = \frac{n}{3}$ with $n \equiv 2 \pmod{3}$.

If s is odd and r is even, then $x_2 = \frac{n}{6}$ with $n \equiv 1 \pmod{6}$.

Examples

r	s	t		a	b_1	b_2	c	x_1	x_2
1	2	3		8	5	3	7	1	2/3
1	3	4		15	7	8	13	3/2	5/3
2	3	5		21	16	5	19	3	7/6
3	4	7		40	33	7	37	6	5/3
1	5	6		35	11	24	31	5/2	14/3
3	5	8		55	39	16	49	15/2	11/3
4	5	9		65	56	9	61	10	13/6
1	6	7		48	13	35	43	3	20/3
5	6	11		96	85	11	91	15	8/3
2	7	9		77	32	45	67	7	55/6
3	7	10		91	51	40	79	21/2	26/3
5	7	12		119	95	24	109	35/2	17/3
6	7	13		133	120	13	127	21	19/6