Arms Race Example in Intro ODEs

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Introduction

- Take-Home Exam
- Algebraic Solution
- Investigating the Algebra
- Different Assumptions

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Elimination Method

Spring 2011

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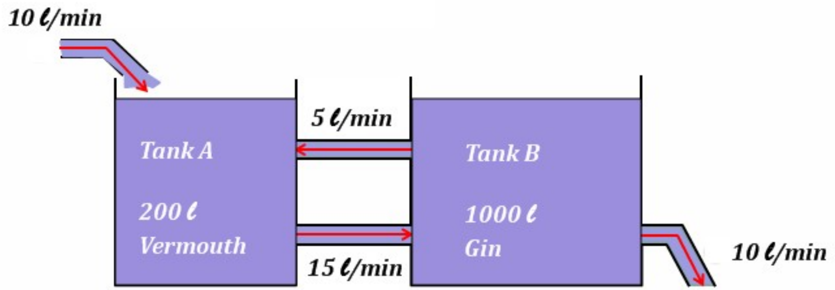

Elimination Method

BuBBy, "Martini Cat" November 18, 2009 via Flickr, Creative Commons Attribution.

Take-Home Exam

An application that would utilize the same techniques but provide a different interpretation.

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If x and y are the expenditures for defense for two countries, suppose that they are governed by:

$$\frac{dx}{dt} = 2y - x + a; \qquad x(0) = 1$$
$$\frac{dy}{dt} = 4x - 3y + b; \qquad y(0) = 4,$$

where a and b are constants that measure the trust (or distrust) each country has for the other. Determine whether there is going to be disarmament (x and y approach 0 as t increases), a stabilized arms race (x and y approach a constant as $t \to +\infty$), or a runaway arms race (x and y approach $+\infty$ as $t \to +\infty$).

Fundamentals of Differential Equations, 7/e. Nagle, Saff, Snider

General Solution

Recall how we can find the solution to such a system via the elimination method:
Just like solving a regular linear system, but now with differential operator D:

$$(D+1)[x]-2[y]=a$$
$$-4[x]+(D+3)[y]=b,$$

where the 1 and 3 are really just shorthand for $1I$ and $3I$ respectively, with I being the identity operator.

General Solution

Then apply $(D+3)$ to the top and $2(I)$ to the bottom, yielding:

$$\begin{aligned} (D+3)(D+1)[x] - 2(D+3)[y] &= (D+3)[a] \\ -8[x] + 2(D+3)[y] &= 2b. \end{aligned}$$

Add the two equations. The result,

$$(D^2 + 4D - 5)[x] = 3a + 2b,$$

is now a second-order, constant-coefficient, non-homogeneous ODE, which students in the class immediately solve.

General Solution

So, with

$$x = c_1 e^{-5t} + c_2 e^{t} - \frac{3a + 2b}{5},$$

we back-substitute into the first ODE and solve for y as well:

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we back-substitute into the first ODE and solve for y as well:

$$y = -2c_1 e^{-5t} + c_2 e^{t} - \frac{4a + b}{5}.$$

Particular Solution

The problem gave us some initial conditions, so applying the fact that $x(0) = 1$ and $y(0) = 4$ yields the answer:

$$x(t) = \left(-1 + \frac{b}{15} - \frac{a}{15}\right) e^{-5t} + \left(2 + \frac{2a}{3} + \frac{b}{3}\right) e^{t} - \frac{3a + 2b}{5}$$
$$y(t) = \left(2 - \frac{2b}{15} + \frac{2a}{15}\right) e^{-5t} + \left(2 + \frac{2a}{3} + \frac{b}{3}\right) e^{t} - \frac{4a + b}{5},$$

which is ripe for interpretation.

Book's Assumptions: "Type" of Limits Must Agree

Under what conditions could both defense budgets go to zero?
Recall: Determine whether there is going to be disarmament (x and y approach 0 as t increases), a stabilized arms race (x and y approach a constant as $t \to +\infty$), or a runaway arms race (x and y approach $+\infty$ as $t \to +\infty$).

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Under what conditions could both defense budgets go to zero?

Answer: None. First note: since they are budgets, x and y must be non-negative. If both budgets converge to zero (or any finite number), then we must have

$$2 + \frac{2a}{3} + \frac{b}{3} = 0.$$

In this case, the budgets would converge to $x_\infty = -\frac{3a+2b}{5}$ and $y_\infty = -\frac{4a+b}{5}$. If both defense budgets go to zero, then the last two equations give that $a = b = 0$, which contradicts the first equation.

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We solve the first equation for a in terms of b and substitute into the formulas for the limits of x and y to yield:

$$x_\infty = \frac{18 - b}{10}$$
$$y_\infty = \frac{b + 12}{5}.$$

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$$x_\infty = \frac{18 - b}{10}$$
$$y_\infty = \frac{b + 12}{5}.$$

Since $x, y \geq 0$, we get ranges on the trust levels for which valid solutions can occur, namely, $b \in [-12, 18]$ (and as a consequence, $a \in [-12, 3]$).

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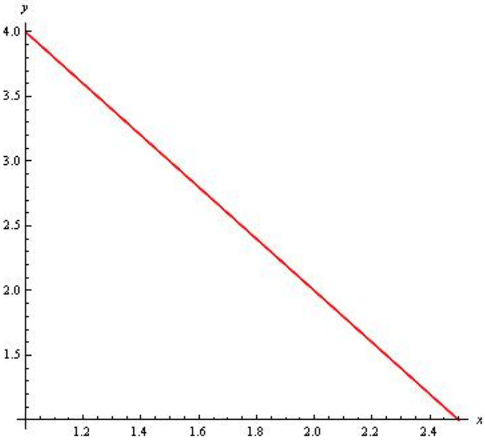
Phase Portraits Examples:



$$(a, b) = \left(\frac{1}{2}, -7\right)$$

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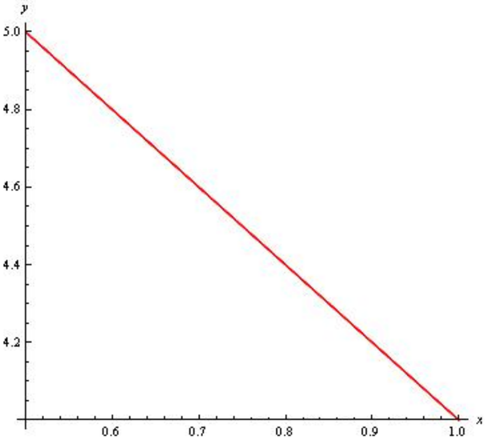

$$(a, b) = \left(-\frac{9}{2}, 13\right)$$

Under what conditions could both defense budgets go to a positive finite number?

Given these ranges of a, b, we might ask at what trust levels is, say, x outspending y? Note that at $a = b = -2$, we get that $x = y = 2$. For $b > -2$, $y > x$. This corresponds to their trust levels, since if $b > -2$, then $a < -2$ (i.e., $b > a$).

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- x and $y \geq 0$ and $2 + \frac{2a}{3} + \frac{b}{3} > 0$

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Solving as before, we see that for any a, b such that $a > -\frac{b}{2} - 3$, the solutions will diverge.

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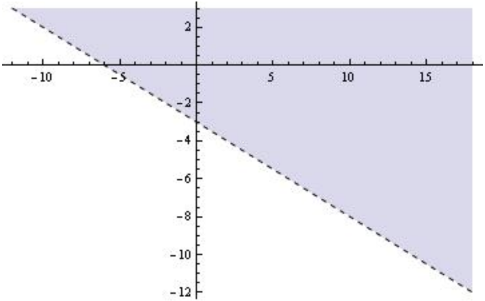

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Must both x and y have the same type of limit (finite vs. infinite)?
Answer: Yes. The decaying term goes to 0 as time increases, and the coefficient for the exponential term matches, which means they are either both staying finite in the long-run or both blowing up (pun intended).

Assumption 1: Same kind of limit

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If $(a, b) = (-12, 18)$, then $x_\infty = 0$ and $y_\infty = 6$.

If $(a, b) = (3, -12)$, then $x_\infty = 3$ and $y_\infty = 0$.

Assumption 2: Meaning of trust

Why are any of the terms there?

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- Other country: higher defense spending in the other country means an increase in defense spending.
- Trust factor...several interpretations to explore, but: negative values of a and b mean that there's a decrease in defense spending, so negative values correspond to trust (and the higher $|a|$ is, the larger the decrease).

Richardson, Arms and Insecurity, (1960)

Assumption 2: Meaning of trust

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What do *a* and *b* even mean?

- Same sign?
- "Unitize" trust?

What if a and b must be of the same sign?

We have the relationship that governs when convergence occurs, namely that $a = -\frac{b}{2} - 3$. If we also require that either $a, b \geq 0$ or $a, b \leq 0$, then a little algebra shows that $b \in [-6, 0]$ (and $a \in [-3, 0]$).

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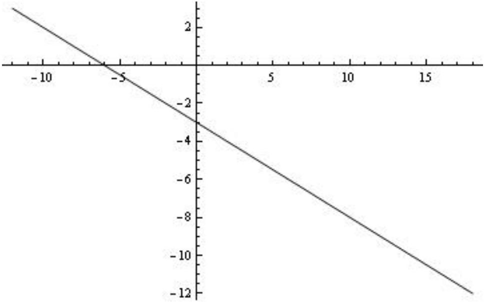

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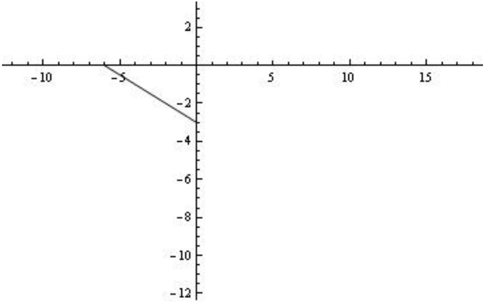

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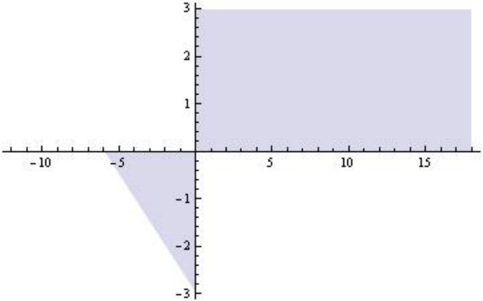
Geometrically, if we're only concerned with mutual types of trust, then our region of divergence becomes:



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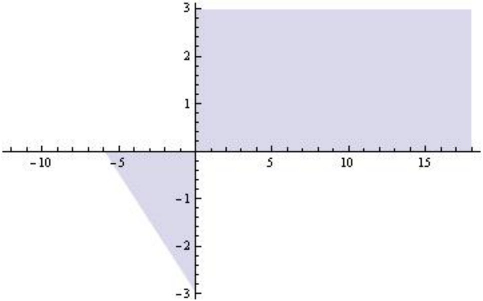

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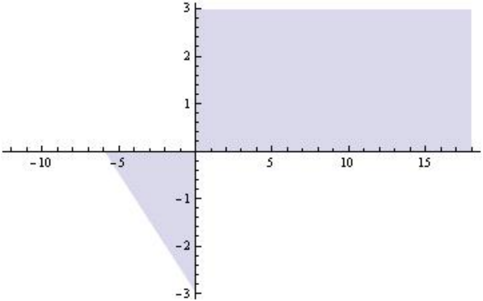

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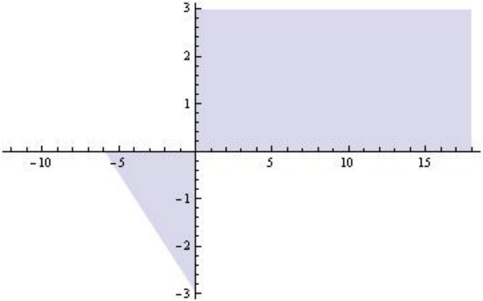

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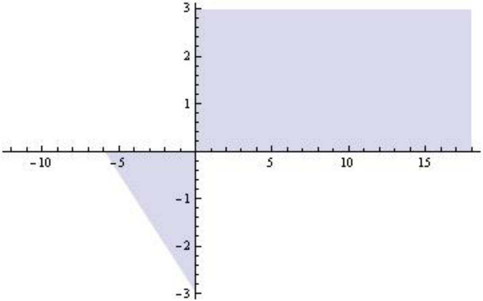

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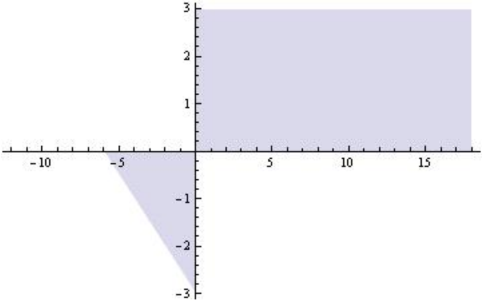

So:

- mutual trust can breed stabilized defense budgets.
- mutual trust can lead to an uncontrolled arms race (!!)
- any mutual distrust will lead to an uncontrolled arms race.

Scales of trust

How does the model react with “unitized” trust values?
Two attempts:

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Two attempts:

- Percentage of trust ($a, b \in [0, 1]$), with 0 being perfect trust.
- Signed percentage of trust ($a, b \in [-1, 1]$), with 0 being neutrality, -1 being perfect trust, 1 being perfect distrust.

Percentage of trust ($a, b \in [0, 1]$)

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In this case, the situation simplifies quite quickly: convergence to a finite number is impossible, and both defense budgets must diverge, since if $b \in [0, 1]$, then any $a \in [0, 1]$ has the property that $a > -\frac{b}{2} - 3$.

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Implication: perfect trust still yields exploding defense budgets.

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Attacking the problem geometrically, we see that in the square $[-1, 1] \times [-1, 1]$ in the *ba*-plane, there is an intersection only with the region of mutual divergence as well.

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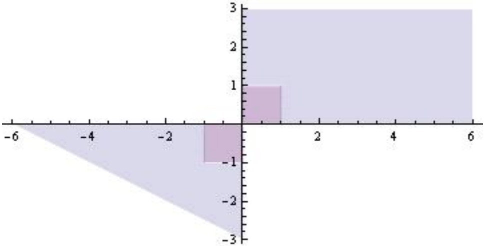

Implication: as before, nothing can prevent defense budget divergence.

When Arms Race, It's Hands Down a Tough Contest

Three Benefits

- Construction/meaning of parameters

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- Construction/meaning of parameters
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- Practical consequences of long-term analysis

When Arms Race, It's Hands Down a Tough Contest

Further Exploration

- Although $a > -\frac{b}{2} - 3$ yields an uncontrolled arms race, the behavior of the solution curves has the potential to violate the fact that x and y must be non-negative. What regions in the ba-plane yield solutions x, y that remain non-negative $\forall t \geq 0$?

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- What if a, b are functions of x, y?
- Vary the coefficients of x and y.

Any questions?

Thank you.