Two Musical Orderings

Messiaen 7

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Partial Orders in Music Theory

• Submajorization and Voice Leading

• Set Inclusion and Pitch Class Sets
  ▸ Straus (2005)

• New Ideas
  ▸ External elements and harmony
  ▸ Stochastic dominance and timbre
Some Connections

- Submajorization and Voice Leading (Tymoczko)
- Orbifolds and Musical Geometry (Tymoczko)
- The Geometry and Topology of Three-Manifolds (Thurston, 1980)
- Geometrization Conjecture (Thurston, 1982)
- Perelman’s proof of the Poincare Conjecture (Perelman, 2003)
- Perelman refuses Fields Medal (2006) and Clay Millenium Prize (Perelman, 2010, $10^6$)
Orderings

• A partial order on a set $S$ is a relation $\leq$ that is

  ▶ Reflexive: $a \leq a$ for all $a \in S$

  ▶ Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in S$

  ▶ Antisymmetric: $a \leq b$ and $b \leq a$ implies $a = b$ for all $a, b \in S$
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![Diagram of a partial order with elements $a$, $b$, and $S$]
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$a, b$ incomparable
Set Inclusion Ordering

• Partial order induced by set inclusion.

• Music theory: scales and harmony
  ‣ C pentatonic: \{C, D, E, G, A\}
  ‣ C diatonic: \{C, D, E, F, G, A, B\}
  ‣ C pentatonic $\subseteq$ C diatonic

• Theme: partial order models some notion of size or precedence among musical objects
Partial Order modulo Group

• Often we want to identify certain musical objects as being essentially “the same”:
  
  ▶ All diatonic scales are “the same”: C major ≡ G major ≡ ...
    
    ▶ “transpositional equivalence”
  
  ▶ All notes separated by whole octaves are “the same”: middle A ≡ high A ≡ ...
    
    ▶ “octave equivalence”
Example: Pitch Class Space

- \( S = \mathbb{Z} \) = “infinite keyboard”
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Partial Order modulo Group

• We may also want to model some notion of musical “motion” or “transformation”:
  ▶ Transpose all notes up one octave.
  ▶ Move from the tonic to the dominant.
Partial Order modulo Group

• A group $G$ acting on $S$ can serve both purposes:
  
  ▸ Equivalence: $a, b \in S$ are “the same” if $b = Ta$ for some $T \in G$.
  
  ▸ Motion: can “move” from $a$ to $b$ if $b = Ta$ for some $T \in G$.
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How do the group and the partial order interact?
Partial Order modulo Group

- Equivalence classes
  
  - $A = [a] = \text{set of all } b \in S \text{ that are essentially “the same” as } a.$
    
    $$= \{ b \in S : b = Ta \text{ for some } T \in G \}$$
  
  - $S / G = \text{set of all distinct equivalence classes}.$
Partial Order modulo Group

• Equivalence classes
  \[ A = [a] = \text{set of all } b \in S \text{ that are essentially “the same” as } a. \]
  \[ = \{ b \in S : b = Ta \text{ for some } T \in G \} \]
  \[ S / G = \text{set of all distinct equivalence classes.} \]

• Induced relation on \( S / G \)
  \[ A \leq B \text{ if and only for all } x \in A \text{ there exists } y \in B \text{ such that } x \leq y \]
Partial Order modulo Group

**Theorem 1**

If $G$ acts *transversely* on $S$, then the induced relation is a partial order on $S / G$. 

*G acts transversely* on $S$.  


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- $G$ acts transversely on $S$ if, for all $T \in G$ and all $a \in S$, either $Ta$ and $a$ are incomparable, or they are identical.
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\[ a \quad Ta \quad a = Ta \]
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A Partial Order on Scales

- *Dense Scale*: a scale consisting solely of steps of size 1 or 2.
A Partial Order on Scales

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\[(2, 2, 1, 2, 2, 2, 1)\]
A Partial Order on Scales

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**Diatonic**: \((2, 2, 1, 2, 2, 2, 1)\)

**Melodic Minor**: \((2, 1, 2, 2, 2, 2, 1)\)

**Octatonic**: \((1, 2, 1, 2, 1, 2, 1, 2)\)
A Partial Order on Scales

- Equivalences
  - Octave equivalence
  - Transpositions (translations) of a scale are all “the same”
    - C major \equiv G major \equiv D major \equiv ...
  - Modes (rotations) of a scale are all “the same”
    - (2,2,1,2,2,2,1) \equiv (2,1,2,2,2,1,2) \equiv (1,2,2,2,1,2,2) \equiv ...
A Partial Order on Scales

- $S = \{\text{all dense scales}\}, \quad \leq = \text{set inclusion}$
A Partial Order on Scales

- $S = \{\text{all dense scales}\}$, $\leq = \text{set inclusion}$

- $G = \langle \text{octave equivalence, transpositions, rotations} \rangle$
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✓ $G$ acts transversely on $S$
A Partial Order on Scales

- $S = \{\text{all dense scales}\}$, $\leq$ = set inclusion
- $G = \langle \text{octave equivalence, transpositions, rotations} \rangle$

$G$ acts transversely on $S$

$S / G$ is a partial order
A Partial Order on Scales

- $S/G$ contains 31 distinct scales.

- $S/G$ has four *minimal elements*:
  - whole tone scale
  - diatonic scale
  - melodic minor scale
  - octatonic scale
A Partial Order on Scales

• Partial order from the whole tone scale
A Partial Order on Scales

- Partial order from the diatonic scale
A Partial Order on Scales

- Partial order from the melodic minor scale

Diagram:

- Melodic minor
- Minor blues (2)
- Messiaen 7
- Chromatic
A Partial Order on Scales

• Partial order from the octatonic scale
A Partial Order on Scales

- Generalization: \( \text{Dense}(k) \text{Scale} \): a scale consisting solely of steps of size 1, or 2, or ... or \( k \).

- \( \text{Dense}(3) \): \((2, 2, 3, 2, 3)\)
- \( \text{Dense}(4) \): \((3, 4, 3, 2)\)
- \( \text{Dense}(5) \): \((4, 3, 5)\)

\[ \begin{array}{c}
\text{pentatonic} \\
(2, 2, 3, 2, 3) \\
\text{Dense}(3)
\end{array} \quad \begin{array}{c}
\text{minor seventh chord} \\
(3, 4, 3, 2) \\
\text{Dense}(4)
\end{array} \quad \begin{array}{c}
\text{major triad} \\
(4, 3, 5) \\
\text{Dense}(5)
\end{array} \]
Theorem 2

A scale in Dense$(k)$ mod $G$ is minimal if and only if every scalar third spans at least $k+1$ semitones.
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- A *scalar third* is the sum of two consecutive steps in a scale.
  - $(2,2,1,2,2,2,1)$ in Dense(2) mod $G$ has scalar thirds $(4,3,3,4,4,3,3)$. 
Theorem 2

A scale in Dense($k$) mod $G$ is minimal if and only if every scalar third spans at least $k + 1$ semitones.

- A **scalar third** is the sum of two consecutive steps in a scale.
  - (2,2,1,2,2,2,1) in Dense(2) mod $G$ has scalar thirds (4,3,3,4,4,3,3).
- Theorem 2 is true in $N$-tone equal temperament.
A Partial Order on Scales

- Dense(3) has seven minimal elements:
  - (1,3,1,3,1,3), a symmetric scale (R. Daly, “Pulp Fiction”)
  - (2,2,2,2,2,2), the whole tone scale
  - (2,2,3,2,3), the pentatonic scale
  - (2,2,3,3,2), the dominant ninth chord
  - (3,1,3,2,3), a blues scale
  - (3,1,3,3,2), the dominant seventh + sharp ninth chord (J. Hendrix, “Foxy Lady”)
  - (3,3,3,3), the fully diminished chord
A Partial Order on Scales

- Dense(4) has an additional six minimal elements:
  - (3,3,4,2), the minor seventh, flat fifth chord
  - (3,4,3,2), the minor seventh chord
  - (4,2,4,2), the dominant seventh, flat fifth chord
  - (4,3,3,2), the dominant seventh chord
  - (4,3,4,1), the major seventh chord
  - (4,4,4), the augmented triad
A Partial Order on Scales

- Dense(5) has an additional four minimal elements:
  - (3,4,5), the minor triad
  - (4,3,5), the major triad
  - (5,1,5,1), a symmetric chord
  - (5,5,2), the quartal triad (H. Hancock, “Maiden Voyage”)
A Timbral Partial Order

• Timbre is the “characteristic sound” of a musical voice.
  ▸ many aspects; notoriously difficult to quantify

• But musicians commonly speak about timbre in comparative ways
  ▸ “a trumpet is brighter than a french horn”
  ▸ “he sings like Bob Dylan with a head cold”

• Can we model these judgements using a partial order?
A Timbral Partial Order

- Discrete power spectrum model for “steady-state timbre”.

\[ a_k = \text{power at } k^{\text{th}} \text{ harmonic} \]
A Timbral Partial Order

- Discrete power spectrum model for "steady-state timbre".

\[ \sum a_k = 1 \text{ (unit volume)} \]
A Timbral Partial Order

- Discrete power spectrum model for “steady-state timbre”.

\[ a = (a_1, a_2, \ldots, a_n) = \text{timbral vector} \]
A Timbral Partial Order

• Discrete power spectrum model for “steady-state timbre”.

\[ S = \{ \text{all timbral vectors} \} = \{ \text{all probability vectors} \} \]
A Timbral Partial Order

• “Brightness” aspect of timbre
  ▸ refers to a prevalence of high harmonics in the sound

• The “Brightness” partial order
  ▸ Timbral vector $b$ is “brighter than” timbral vector $a$ if

$$\sum_{j \geq k} a_j \leq \sum_{j \geq k} b_j \quad \forall k$$

  ▸ i.e. every high-pass filter returns more power from $b$ than from $a$
A Timbral Partial Order

Six Instruments in the “Brightness” Order
Sound Design Problem

• Among all instruments which are no brighter than a trumpet, which has the timbre that is closest to an oboe?

• How do we measure “closeness”?

• Total Variational Distance

\[ d_{TV}(x, y) = \left\{ \sum_{i \in I} |x_i - y_i| : I \subseteq 1, 2, \ldots, n \right\} \]

▷ maximum power differential across subsets of harmonics
Sound Design Problem

- Constrained Optimization Problem

Minimize: $d_{TV}(x, \text{ oboe})$
Subject to: $x \leq \text{ trumpet in the “brightness” order}$
Sound Design Problem

- Constrained Optimization Problem

Minimize: $\|x - \text{oboe}\|_1$
Subject to: $Hx \leq H(\text{trumpet})$ component-wise
Sound Design Problem

- Constrained Optimization Problem

Minimize: $\| x - \text{oboe} \|_1$

Subject to: $H x \leq H(\text{trumpet})$ component-wise

- efficiently solvable via linear programming
Sound Design Problem

Minimize: $\|x - oboe\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise
Sound Design Problem

Minimize: $\|x - oboe\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise
Sound Design Problem

Minimize: $\|x - \text{oBoe}\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise

Solution to Sound Design Problem
Sound Design Problem

Minimize: $\|x - \text{ooboe}\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise
Sound Design Problem

Minimize: $\|x - \text{ooboe}\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise

Solution to Sound Design Problem
Sound Design Problem

Minimize:  $\| x - oboe \|_1$

Subject to:  $Hx \leq H(\text{trumpet})$ component-wise

Solution to Sound Design Problem

Thanks!
Some References

• D. Tymoczko, *The geometry of musical chords*, Science 313, 2006, pp. 72 - 74; available at http://dx.doi.org/10.1126/science.1126287

