AN ELEMENTARY PROOF OF THE MEAN INEQUALITIES

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Abstract

In this paper our goal is to prove the well-known chain of inequalities involving the harmonic, geometric, logarithmic, identric, and arithmetic means using nothing more than basic calculus.
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Of course, these results are all well-known and several proofs of them and their generalizations have been given:
Hardy, Littlewood, and Polya (1964)
Carlson (1965)
Carlson and Tobey (1968)
Beckenbach and Bellman (1971)
Alzer (1985a, 1985b)
Abstract

Our purpose here is to present a unified approach and give the proofs as corollaries of one elementary theorem.
The Pythagorean Means

For a sequence of numbers \( x = \{x_1, x_2, \ldots, x_n\} \) we will let

\[
AM(x_1, x_2, \ldots, x_n) = AM(x) = \frac{\sum_{j=1}^{n} x_j}{n}
\]

\[
GM(x_1, x_2, \ldots, x_n) = GM(x) = \prod_{j=1}^{n} x_j^{1/n}
\]

and

\[
HM(x_1, x_2, \ldots, x_n) = HM(x) = \frac{n}{\sum_{j=1}^{n} \frac{1}{x_j}}
\]

to denote the well-known arithmetic, geometric, and harmonic means, also called the Pythagorean means (PM).
Pythagorean Means

The Pythagorean means have the obvious properties:

1. \( PM(x_1, x_2, \ldots, x_n) \) is independent of order

2. \( PM(x, x, \ldots, x) = x \)

3. \( PM(bx_1, bx_2, \ldots, bx_n) = bPM(x_1, x_2, \ldots, x_n) \)
1. 

2. $PM(x_1, x_2)$ is always a solution of a simple equation. In particular, the arithmetic mean of two numbers $x_1$ and $x_2$ can be defined via the equation

$$AM - x_1 = x_2 - AM$$

The harmonic mean satisfies the same relation with reciprocals, that is, it is a solution of the equation

$$\frac{1}{x_1} - \frac{1}{HM} = \frac{1}{HM} - \frac{1}{x_2}$$

The geometric mean of two numbers $x_1$ and $x_2$ can be visualized as the solution of the equation

$$\frac{x_1}{GM} = \frac{GM}{x_2}$$
1. \[ GM = \sqrt{(AM)(HM)} \]

2. \[ HM \left( x_1, \frac{1}{x_1} \right) = \frac{1}{AM(x_1, \frac{1}{x_1})} \]

3. \[ (x_1 + x_2 + \cdots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right) \geq n^2 \]

This follows because

\[ \frac{AM(x_1, x_2, \ldots, x_n)}{HM(x_1, x_2, \ldots, x_n)} \geq 1 \]
Logarithmic and Identicric Means

\[ LM(0, x_2) = LM(x_1, 0) = 0 \]

\[ LM(x_1, x_1) = x_1 \]

and for positive distinct numbers \( x_1 \) and \( x_2 \)

\[ LM(x_1, x_2) = \frac{x_2 - x_1}{lnx_2 - lnx_1} \]
Logarithmic and Identric Means

The following are some basic properties of the logarithmic means:

Logarithmic mean $LM(a, b)$ can be thought of as the mean-value of the function $f(x) = \ln x$ over the interval $[a, b]$. 
The logarithmic mean can also be interpreted as the area under an exponential curve.

Since

\[
\int_0^1 x^{1-t} y^t \, dt = x \int_0^1 \left(\frac{y}{x}\right)^t \, dt = \frac{x - y}{\ln x - \ln y}
\]

We also have the identity

\[LM(x, y) = \int_0^1 x^{1-t} y^t \, dt\]

Using this representation it is easy to show that

\[LM(cx, cy) = cLM(x, y)\]
Logarithmic and Identric Means

We have the identity

\[
\frac{LM(x^2, y^2)}{LM(x, y)} = AM(x, y)
\]

which follows easily:

\[
\frac{LM(x^2, y^2)}{LM(x, y)} = \frac{x^2 - y^2}{2(lnx - lny)} \div \frac{x - y}{lnx - lny} = \frac{x + y}{2}
\]
The *identric mean* of two distinct positive real numbers $x_1, x_2$ is defined as:

$$IM(x_1, x_2) = \frac{1}{e} \left( \frac{x_2}{x_1} \right)^{\frac{1}{x_2-x_1}}$$

with

$$IM(x_1, x_1) = x_1$$
The slope of the secant line joining the points \((a, f(a))\) and \((b, f(b))\) on the graph of the function 

\[ f(x) = x \ln(x) \]

is the natural logarithm of \(IM(a, b)\).
Theorem 1. Suppose $f : [a, b] \to \mathbb{R}$ is a function with a strictly increasing derivative. Then

\[
\int_a^b f(t)dt < \frac{b - a}{2} \left[ f(s) - s \left( \frac{f(b) - f(a)}{b - a} \right) + \frac{bf(b) - af(a)}{b - a} \right]
\]

for all $s < t$ in $[a, b]$. 
Let $s_0$ be defined by the equation

$$f'(s_0) = \frac{f(b) - f(a)}{b - a}$$

Then,

$$\int_a^b f(t)dt < \frac{b - a}{2} \left[ f(s_0) - s_0 \left( \frac{f(b) - f(a)}{b - a} \right) ight] + \frac{bf(b) - af(a)}{b - a}$$

is the sharpest form of the above inequality.
Proof. By the Mean Value Theorem, for all $s, t$ in $[a, b]$, we have

$$\frac{f(t) - f(s)}{t - s} < f'(u)$$

for some $u$ between $s$ and $t$. Assuming without loss of generality $s < t$, by the assumption of the theorem we have

$$f(t) - f(s) < (t - s)f'(t)$$
Integrating both sides with respect to $t$, we have
\[
\int_a^b f(t)\,dt < (b - a)f(s) + bf(b) - af(a) - s[f(b) - f(a)] - \int_a^b f(t)\,dt
\]
and the inequality of the theorem follows.
The Main Theorem

Let us now put

\[ g(s) = (b - a)f(s) + bf(b) - af(a) \]

\[ - s[f(b) - f(a)] - \int_a^b f(t)dt \]

Note that

\[ g'(s) = (b - a)f'(s) - [f(b) - f(a)] \]
Moreover, since

\[ g(b) = g(a) = (b - a)[f(a) + f(b)] - \int_a^b f(t)dt \]

there exists an \( s_0 \) in \((a, b)\) such that \( g'(s_0) = 0 \).
Since $f'$ is strictly increasing, we have
\[ g'(s) > g'(s_0) = 0 \]
for $s > s_0$
and
\[ g'(s) < g'(s_0) = 0 \]
for $s < s_0$

Thus, $s_0$ is a minimum of $g$ and $g(s_0) \leq g(s)$ for all $s$ in $[a, b]$. 

**The Main Theorem**
Proof of the Mean Inequalities

Let us now assume that $0 < a < b$.

Let us let $f(t) = \frac{1}{t^2}$. The condition of the Theorem 1 is satisfied.

We compute

$$s_0 = (abH)^{1/3}$$
Proof of the Mean Inequalities

And the inequality of the theorem becomes

\[ HM(a, b) < GM(a, b) \]

Now, let us let \( f(t) = \frac{1}{t} \). The condition of Theorem 1 is satisfied.
Proof of the Mean Inequalities

One can easily compute

\[ s_0 = \sqrt{ab} \]

And the inequality of the theorem becomes

\[ GM(a, b) < LM(a, b) \]
Proof of the Mean Inequalities

Now let $f(t) = -\ln t$. Again the condition of Theorem 1 is satisfied. The $s_0$ of the theorem can be computed from the equation

$$s_0 = L$$

the logarithmic mean of $a$ and $b$. 
Proof of the Mean Inequalities

The inequality of the theorem becomes

\[ LM(a, b) < IM(a, b) \]
Finally, let us put $f(t) = tlnt$. Again the condition of Theorem 1 is satisfied.

In this case,

$$s_0 = I$$

The identric mean of a and b.
Proof of the Mean Inequalities

The inequality of the theorem becomes

\[ IM(a, b) < AM(a, b) \]
Proof of the Mean Inequalities

Thus, we now have for $a \neq b$

$$HM(a, b) < GM(a, b) < LM(a, b) < IM(a, b) < AM(a, b)$$

Alzer H. 1985b. Ungleichungen für \((e/a)^a (b/e)^b\). In Elemente Math., 40 (1985), p. 120-123.

