The symmetries of $(7, 3, 1)$

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What To Expect

The $(7, 3, 1)$ design

Symmetries of a design

The Orbit-Stabilizer Theorem

The number of symmetries of $(7, 3, 1)$

A surprise ending

The $(7,3,1)$ block design

$$\boxed{\begin{array}{l} B_1 = \{1,2,4\} \\ B_2 = \{2,3,5\} \\ B_3 = \{3,4,6\} \\ B_4 = \{4,5,7\} \\ B_5 = \{5,6,1\} \\ B_6 = \{6,7,2\} \\ B_7 = \{7,1,3\} \end{array}}$$

The $(7,3,1)$ block design is:

- a set V of 7 items (or *varieties*) and a collection of 7 subsets of V called *blocks*, such that
- each block contains three varieties,
- each variety is in three blocks, and
- each pair of varieties is in exactly one block together.

Permutations and the cycle notation

- A *permutation* of a set S is a mapping of S to itself that is one-to-one and onto.

- The permutation $\rho : 1 \to 2 \to 4 \to 1, 3 \to 6 \to 5 \to 3, 7 \to 7$ of the set $\{1, 2, 3, 4, 5, 6, 7\}$ can also be written as $\rho = (1, 2, 4)(3, 6, 5)(7)$.

Symmetries of block designs

- A *symmetry of a design* is a permutation of its varieties that also permutes its blocks.

- The permutation $\rho = (1, 2, 4)(3, 6, 5)(7)$ determines (or *induces*) the permutation ρ^* : $(B_1)(B_2, B_3, B_5)(B_4, B_7, B_6)$ on the blocks of $(7, 3, 1)$. Thus, ρ is a symmetry of $(7, 3, 1)$.

- The symmetries of a design $\mathcal{D}$ form a group under composition of mappings — the symmetry group $Sym(\mathcal{D})$.

Orbits and Stabilizers

Let G be a group of permutations of a set S, let $T \subseteq S$ and let $x \in S$.

- $Orb_G(x) = \{y \in S : y = g(x) \text{ for some } g \in G\}$ is called the *orbit of x under G*. Similarly, $Orb_G(T) = \{R \subseteq S : R = g(T) \text{ for some } g \in G\}$ is called the orbit of T under G.

- $Stab_G(T) = \{g \in G : g(T) = T\}$ is called the *stabilizer* of T in G; we denote $Stab_G(\{x\})$ by $Stab_G(x)$. If $\alpha \in Stab_G(T)$, we say that α *stabilizes* or *fixes* T.

The Orbit-Stabilizer Theorem

Theorem: Let G be a finite group of permutations on a finite set S, let $T \subseteq S$, and let $|X|$ denote the cardinality (or order) of X. Then $Stab_G(T)$ is a subgroup of G, and the cardinalities of G, $Stab_G(T)$, and $Orb_G(T)$ are related by the equation

$$|G| = |Orb_G(T)| \cdot |Stab_G(T)|.$$

Some symmetry groups and how they are related

Let $\mathcal{D}$ denote the $(7, 3, 1)$ design. We define the following groups:

- $G = Sym(\mathcal{D})$ – the symmetry group of $\mathcal{D}$
- $H = Stab_G(B_1)$ – the symmetries that fix B_1
- $K = Stab_H(1)$ – the symmetries that fix B_1 and 1
- $L = Stab_K(2)$ – the symmetries that fix B_1 and 1 and 2

By three applications of the Orbit-Stabilizer Theorem,

$$|G| = |Orb_G(B_1)| \cdot |Orb_H(1)| \cdot |Orb_K(2)| \cdot |L|.$$

The symmetry τ

Define τ on $\{1,2,3,4,5,6,7\}$ by $\tau = (1,2,3,4,5,6,7)$. **Fact:** $\tau \in G$.

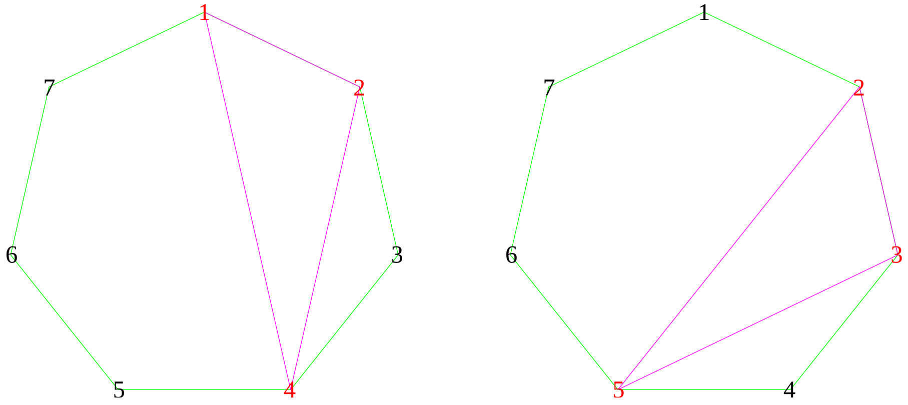
Here's how the induced map τ^* acts on the blocks of $(7,3,1)$:

$$\begin{aligned}
\tau(B_1) &= \tau(\{1,2,4\}) = \{2,3,5\} = B_2 \\
\tau(B_2) &= \tau(\{2,3,5\}) = \{3,4,6\} = B_3 \\
\tau(B_3) &= \tau(\{3,4,6\}) = \{4,5,7\} = B_4 \\
\tau(B_4) &= \tau(\{4,5,7\}) = \{5,6,1\} = B_5 \\
\tau(B_5) &= \tau(\{5,6,1\}) = \{6,7,2\} = B_6 \\
\tau(B_6) &= \tau(\{6,7,2\}) = \{7,1,3\} = B_7 \\
\tau(B_7) &= \tau(\{7,1,3\}) = \{1,2,4\} = B_1
\end{aligned}$$

Thus, $\tau^* = (B_1, B_2, B_3, B_4, B_5, B_6, B_7)$, so $|Orb_G(B_1)| = 7$.

τ and τ^*

$$\tau^* = (B_1, B_2, B_3, B_4, B_5, B_6, B_7)$$



$\tau = (1, 2, 3, 4, 5, 6, 7)$ cyclically permutes $1, 2, 3, 4, 5, 6$ and 7.

The symmetry ρ

Define ρ on $\{1,2,3,4,5,6,7\}$ by $\rho = (1,2,4)(3,6,5)(7)$. **Fact:** ρ fixes B_1, so $\rho \in H = Stab_G(B_1)$.

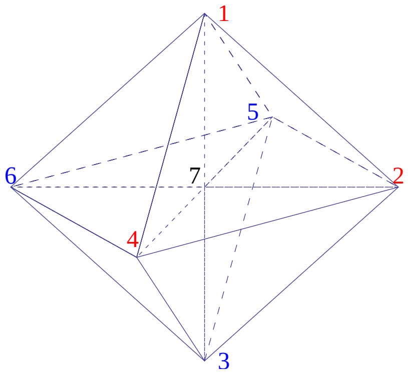
Here's how the induced map ρ^* acts on the blocks of $(7,3,1)$:

$$\begin{aligned}
\rho(B_1) &= \rho(\{1,2,4\}) = \{2,4,1\} = B_1\\
\rho(B_2) &= \rho(\{2,3,5\}) = \{4,6,3\} = B_3\\
\rho(B_3) &= \rho(\{3,4,6\}) = \{6,1,5\} = B_5\\
\rho(B_4) &= \rho(\{4,5,7\}) = \{1,3,7\} = B_7\\
\rho(B_5) &= \rho(\{5,6,1\}) = \{3,5,2\} = B_2\\
\rho(B_6) &= \rho(\{6,7,2\}) = \{5,7,4\} = B_4\\
\rho(B_7) &= \rho(\{7,1,3\}) = \{7,2,6\} = B_6
\end{aligned}$$

Thus, $\rho^* = (B_1)(B_2,B_3,B_5)(B_4,B_7,B_6)$; as ρ cyclically permutes $1,2$ and 4, we see that $|Orb_H(1)| = 3$.

ρ and ρ^*

$$\rho^* = (B_1)(B_2, B_3, B_5)(B_4, B_7, B_6)$$



$\rho = (1,2,4)(3,6,5)(7)$ rotates $\{1,2,4\}$, rotates $\{3,6,5\}$, fixes 7.

The symmetry σ

Define σ on $\{1,2,3,4,5,6,7\}$ by $\sigma = (1)(2,4)(3,5,7,6)$. **Fact:** σ fixes both B_1 and 1, so $\sigma \in K = Stab_H(1)$.

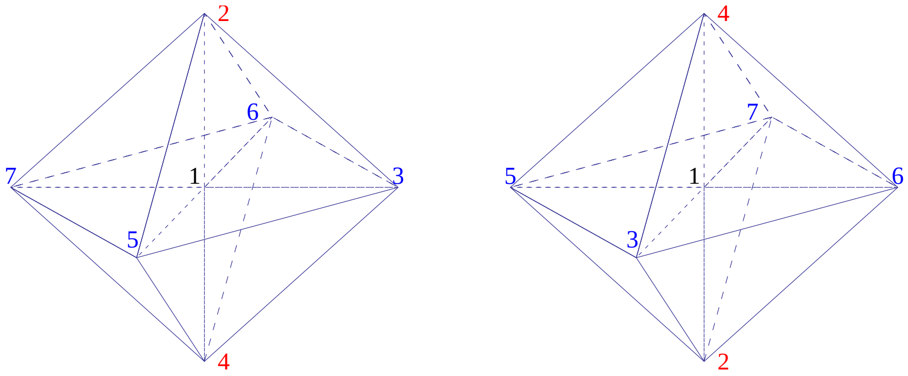
Here's how the induced map σ^* acts on the blocks of $(7,3,1)$:

$$\begin{aligned}
\sigma(B_1) &= \sigma(\{1,2,4\}) = \{1,4,2\} = B_1 \\
\sigma(B_2) &= \sigma(\{2,3,5\}) = \{4,5,7\} = B_4 \\
\sigma(B_3) &= \sigma(\{3,4,6\}) = \{5,2,3\} = B_2 \\
\sigma(B_4) &= \sigma(\{4,5,7\}) = \{2,7,6\} = B_6 \\
\sigma(B_5) &= \sigma(\{5,6,1\}) = \{7,3,1\} = B_7 \\
\sigma(B_6) &= \sigma(\{6,7,2\}) = \{3,6,4\} = B_3 \\
\sigma(B_7) &= \sigma(\{7,1,3\}) = \{6,1,5\} = B_5
\end{aligned}$$

Thus, $\sigma^* = (B_1)(B_2, B_4, B_6, B_3)(B_5, B_7)$. As σ switches 2 and 4, we see that $|Orb_K(2) = 2|$.

σ and σ^*

$$\sigma^* = (B_1)(B_2, B_4, B_6, B_3)(B_5, B_7)$$



$\sigma = (1)(2,4)(3,5,7,6)$ fixes 1, swaps 2 and 4, rotates $\{3,5,7,6\}$.

The symmetry δ

Define δ on $\{1, 2, 3, 4, 5, 6, 7\}$ by $\delta = (1)(2)(4)(3, 5)(6, 7)$. **Fact:** $L = \{identity, \delta, \sigma^2, \delta\sigma^2\}$ – thus, $|Stab_K(2)| = |L| = 4$.

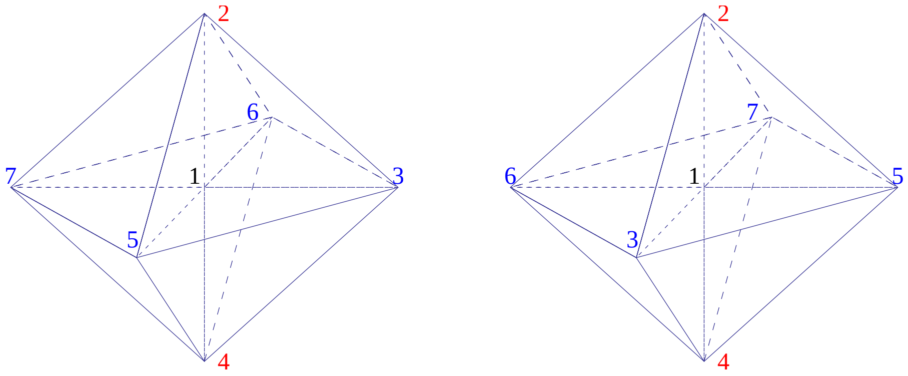
Here's how the induced map δ^* acts on the blocks of $(7, 3, 1)$:

$$\begin{aligned}
\delta(B_1) &= \delta(\{1,2,4\}) = \{1,2,4\} = B_1\\
\delta(B_2) &= \delta(\{2,3,5\}) = \{2,5,3\} = B_2\\
\delta(B_3) &= \delta(\{3,4,6\}) = \{5,4,7\} = B_4\\
\delta(B_4) &= \delta(\{4,5,7\}) = \{4,3,6\} = B_3\\
\delta(B_5) &= \delta(\{5,6,1\}) = \{3,7,1\} = B_7\\
\delta(B_6) &= \delta(\{6,7,2\}) = \{7,6,2\} = B_6\\
\delta(B_7) &= \delta(\{7,1,3\}) = \{6,1,5\} = B_5
\end{aligned}$$

Thus, $\delta^* = (B_1)(B_2)(B_3, B_4)(B_5, B_7)(B_6)$.

δ and δ^*

$$\delta^* = (B_1)(B_2)(B_3, B_4)(B_5, B_7)(B_6)$$



$\delta = (1)(2)(4)(3,5)(7,6)$ fixes $1, 2$ and 4, swaps 3 and 5, swaps 7 and 6.

Determining orbits and stabilizers

- $\tau^* = (B_1, B_2, B_3, B_4, B_5, B_6, B_7) \in G$, so $|Orb_G(B_1)| = 7$.

- $\rho^* = (B_1)(B_2, B_3, B_5)(B_4, B_7, B_6) \in H = Stab_G(B_1)$, and $\rho = (1, 2, 4)(3, 6, 5)(7)$, so $|Orb_H(1)| = 3$.

- $\sigma = (1)(2, 4)(3, 5, 7, 6)$ fixes 1 and swaps 2 and 4, so $\sigma^* = (B_1)(B_2, B_4, B_6, B_3)(B_5, B_7) \in K = Stab_H(1)$, and $|Orb_K(2)| = 2$.

- $\delta = (1)(2)(4)(3, 5)(7, 6)$ fixes 1, 2 and 4, so $\delta^* = (B_1)(B_2)(B_3, B_4)(B_5, B_7)(B_6) \in L = Stab_K(2)$. In fact, $L = \{id, \delta^*, \sigma^{*2}, \delta^*\sigma^{*2}\}$ and $|L| = 4$.

Let's "do the math"

The Orbit-Stabilizer Theorem tells us that for $G = Sym(\mathcal{D})$,

$$\begin{aligned}|G| &= |Orb_G| \cdot |Orb_H(1)| \cdot |Orb_K(2)| \cdot |Stab_K(2)| \\ &= 7 \cdot 3 \cdot 2 \cdot 4 \\ &= 168.\end{aligned}$$

Hence, there are 168 symmetries of $(7, 3, 1)$.

Facts about *Sym*((7,3,1))

- *Sym*((7,3,1)) is generated by $\tau = (1,2,3,4,5,6,7)$ and $\sigma = (1)(2,4)(3,5,7,6)$.
- *Sym*((7,3,1)) is commonly known as $GL(3,2)$, the 3×3 matrices with entries in $\mathbb{Z}$ mod 2.
- Another name for $GL(3,2)$ is $PSL(2,7)$, the 2×2 matrices with entries in $\mathbb{Z}$ mod 7 and determinant 1, with I and $-I$ identified.
- *Sym*((7,3,1)) is *simple*: it has no nontrivial normal subgroups.
- . . . and . . .

The Surprise Ending

The Surprise Ending

Sym((7,3,1)) contains

within its subgroup structure

a copy of the $(7, 3, 1)$ design.

THANK YOU!