The symmetries of $(7, 3, 1)$

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What To Expect

The (7, 3, 1) design

Symmetries of a design

The Orbit-Stabilizer Theorem

The number of symmetries of (7, 3, 1)

A surprise ending
The (7, 3, 1) block design

The (7, 3, 1) block design is:

- a set $V$ of 7 items (or varieties) and a collection of 7 subsets of $V$ called blocks, such that
- each block contains three varieties,
- each variety is in three blocks, and
- each pair of varieties is in exactly one block together.

\[
B_1 = \{1, 2, 4\} \\
B_2 = \{2, 3, 5\} \\
B_3 = \{3, 4, 6\} \\
B_4 = \{4, 5, 7\} \\
B_5 = \{5, 6, 1\} \\
B_6 = \{6, 7, 2\} \\
B_7 = \{7, 1, 3\}
\]
A permutation of a set $S$ is a mapping of $S$ to itself that is one-to-one and onto.

The permutation $\rho : 1 \rightarrow 2 \rightarrow 4 \rightarrow 1, 3 \rightarrow 6 \rightarrow 5 \rightarrow 3, 7 \rightarrow 7$ of the set $\{1, 2, 3, 4, 5, 6, 7\}$ can also be written as $\rho = (1, 2, 4)(3, 6, 5)(7)$. 
A symmetry of a design is a permutation of its varieties that also permutes its blocks.

The permutation \( \rho = (1, 2, 4)(3, 6, 5)(7) \) determines (or induces) the permutation \( \rho^* : (B_1)(B_2, B_3, B_5)(B_4, B_7, B_6) \) on the blocks of \((7, 3, 1)\). Thus, \( \rho \) is a symmetry of \((7, 3, 1)\).

The symmetries of a design \( \mathcal{D} \) form a group under composition of mappings — the symmetry group \( \text{Sym}(\mathcal{D}) \).
Let $G$ be a group of permutations of a set $S$, let $T \subseteq S$ and let $x \in S$.

- $\text{Orb}_G(x) = \{y \in S : y = g(x) \text{ for some } g \in G\}$ is called the orbit of $x$ under $G$. Similarly, $\text{Orb}_G(T) = \{R \subseteq S : R = g(T) \text{ for some } g \in G\}$ is called the orbit of $T$ under $G$.

- $\text{Stab}_G(T) = \{g \in G : g(T) = T\}$ is called the stabilizer of $T$ in $G$; we denote $\text{Stab}_G(\{x\})$ by $\text{Stab}_G(x)$. If $\alpha \in \text{Stab}_G(T)$, we say that $\alpha$ stabilizes or fixes $T$. 

\[ \text{Stab}_G(T) = \{g \in G : g(T) = T\} \] is called the stabilizer of $T$ in $G$; we denote $\text{Stab}_G(\{x\})$ by $\text{Stab}_G(x)$. If $\alpha \in \text{Stab}_G(T)$, we say that $\alpha$ stabilizes or fixes $T$. 

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The Orbit-Stabilizer Theorem

**Theorem:** Let $G$ be a finite group of permutations on a finite set $S$, let $T \subseteq S$, and let $|X|$ denote the cardinality (or order) of $X$. Then $\text{Stab}_G(T)$ is a subgroup of $G$, and the cardinalities of $G$, $\text{Stab}_G(T)$, and $\text{Orb}_G(T)$ are related by the equation

$$|G| = |\text{Orb}_G(T)| \cdot |\text{Stab}_G(T)|.$$
Let $\mathcal{D}$ denote the $(7, 3, 1)$ design. We define the following groups:

- $G = \text{Sym}(\mathcal{D})$ – the symmetry group of $\mathcal{D}$
- $H = \text{Stab}_G(B_1)$ – the symmetries that fix $B_1$
- $K = \text{Stab}_H(1)$ – the symmetries that fix $B_1$ and 1
- $L = \text{Stab}_K(2)$ – the symmetries that fix $B_1$ and 1 and 2

By three applications of the Orbit-Stabilizer Theorem,

$$|G| = |\text{Orb}_G(B_1)| \cdot |\text{Orb}_H(1)| \cdot |\text{Orb}_K(2)| \cdot |L|.$$
Define $\tau$ on $\{1, 2, 3, 4, 5, 6, 7\}$ by $\tau = (1, 2, 3, 4, 5, 6, 7)$. **Fact:** $\tau \in G$.

Here’s how the induced map $\tau^*$ acts on the blocks of $(7, 3, 1)$:

\[
\begin{align*}
\tau(B_1) &= \tau(\{1, 2, 4\}) = \{2, 3, 5\} = B_2 \\
\tau(B_2) &= \tau(\{2, 3, 5\}) = \{3, 4, 6\} = B_3 \\
\tau(B_3) &= \tau(\{3, 4, 6\}) = \{4, 5, 7\} = B_4 \\
\tau(B_4) &= \tau(\{4, 5, 7\}) = \{5, 6, 1\} = B_5 \\
\tau(B_5) &= \tau(\{5, 6, 1\}) = \{6, 7, 2\} = B_6 \\
\tau(B_6) &= \tau(\{6, 7, 2\}) = \{7, 1, 3\} = B_7 \\
\tau(B_7) &= \tau(\{7, 1, 3\}) = \{1, 2, 4\} = B_1
\end{align*}
\]

Thus, $\tau^* = (B_1, B_2, B_3, B_4, B_5, B_6, B_7)$, so $|\text{Orb}_G(B_1)| = 7$. 

**The symmetries of $(7, 3, 1)$**
\[ \tau^* = (B_1, B_2, B_3, B_4, B_5, B_6, B_7) \]

\[ \tau = (1, 2, 3, 4, 5, 6, 7) \] cyclically permutes 1, 2, 3, 4, 5, 6 and 7.

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The symmetries of (7, 3, 1)
Define $\rho$ on \{1, 2, 3, 4, 5, 6, 7\} by $\rho = (1, 2, 4)(3, 6, 5)(7)$. **Fact:** $\rho$ fixes $B_1$, so $\rho \in H = Stab_G(B_1)$.

Here’s how the induced map $\rho^*$ acts on the blocks of (7, 3, 1):

\[
\begin{align*}
\rho(B_1) &= \rho(\{1, 2, 4\}) = \{2, 4, 1\} = B_1 \\
\rho(B_2) &= \rho(\{2, 3, 5\}) = \{4, 6, 3\} = B_3 \\
\rho(B_3) &= \rho(\{3, 4, 6\}) = \{6, 1, 5\} = B_5 \\
\rho(B_4) &= \rho(\{4, 5, 7\}) = \{1, 3, 7\} = B_7 \\
\rho(B_5) &= \rho(\{5, 6, 1\}) = \{3, 5, 2\} = B_2 \\
\rho(B_6) &= \rho(\{6, 7, 2\}) = \{5, 7, 4\} = B_4 \\
\rho(B_7) &= \rho(\{7, 1, 3\}) = \{7, 2, 6\} = B_6
\end{align*}
\]

Thus, $\rho^* = (B_1)(B_2, B_3, B_5)(B_4, B_7, B_6)$; as $\rho$ cyclically permutes 1, 2 and 4, we see that $|Orb_H(1)| = 3$. 
$\rho^* = (B_1)(B_2, B_3, B_5)(B_4, B_7, B_6)$

$\rho = (1, 2, 4)(3, 6, 5)(7)$ rotates \{1, 2, 4\}, rotates \{3, 6, 5\}, fixes 7.
The symmetry $\sigma$

Define $\sigma$ on $\{1, 2, 3, 4, 5, 6, 7\}$ by $\sigma = (1)(2, 4)(3, 5, 7, 6)$. **Fact:** $\sigma$ fixes both $B_1$ and 1, so $\sigma \in K = Stab_H(1)$.

Here’s how the induced map $\sigma^*$ acts on the blocks of $(7, 3, 1)$:

\[
\begin{align*}
\sigma(B_1) &= \sigma(\{1, 2, 4\}) = \{1, 4, 2\} = B_1 \\
\sigma(B_2) &= \sigma(\{2, 3, 5\}) = \{4, 5, 7\} = B_4 \\
\sigma(B_3) &= \sigma(\{3, 4, 6\}) = \{5, 2, 3\} = B_2 \\
\sigma(B_4) &= \sigma(\{4, 5, 7\}) = \{2, 7, 6\} = B_6 \\
\sigma(B_5) &= \sigma(\{5, 6, 1\}) = \{7, 3, 1\} = B_7 \\
\sigma(B_6) &= \sigma(\{6, 7, 2\}) = \{3, 6, 4\} = B_3 \\
\sigma(B_7) &= \sigma(\{7, 1, 3\}) = \{6, 1, 5\} = B_5
\end{align*}
\]

Thus, $\sigma^* = (B_1)(B_2, B_4, B_6, B_3)(B_5, B_7)$. As $\sigma$ switches 2 and 4, we see that $|Orb_K(2) = 2|$. 

Brown  The symmetries of $(7, 3, 1)$
\[ \sigma^* = (B_1)(B_2, B_4, B_6, B_3)(B_5, B_7) \]

\[ \sigma = (1)(2, 4)(3, 5, 7, 6) \text{ fixes 1, swaps 2 and 4, rotates } \{3, 5, 7, 6\}. \]
The symmetry $\delta$

Define $\delta$ on $\{1, 2, 3, 4, 5, 6, 7\}$ by $\delta = (1)(2)(4)(3, 5)(6, 7)$. **Fact:**

$L = \{\text{identity}, \delta, \sigma^2, \delta \sigma^2\}$ – thus, $|\text{Stab}_K(2)| = |L| = 4$.

Here’s how the induced map $\delta^*$ acts on the blocks of $(7, 3, 1)$:

\[
\begin{align*}
\delta(B_1) &= \delta(\{1, 2, 4\}) = \{1, 2, 4\} = B_1 \\
\delta(B_2) &= \delta(\{2, 3, 5\}) = \{2, 5, 3\} = B_2 \\
\delta(B_3) &= \delta(\{3, 4, 6\}) = \{5, 4, 7\} = B_4 \\
\delta(B_4) &= \delta(\{4, 5, 7\}) = \{4, 3, 6\} = B_3 \\
\delta(B_5) &= \delta(\{5, 6, 1\}) = \{3, 7, 1\} = B_7 \\
\delta(B_6) &= \delta(\{6, 7, 2\}) = \{7, 6, 2\} = B_6 \\
\delta(B_7) &= \delta(\{7, 1, 3\}) = \{6, 1, 5\} = B_5
\end{align*}
\]

Thus, $\delta^* = (B_1)(B_2)(B_3, B_4)(B_5, B_7)(B_6)$. 
\[ \delta^* = (B_1)(B_2)(B_3, B_4)(B_5, B_7)(B_6) \]

\[ \delta = (1)(2)(4)(3, 5)(7, 6) \] fixes 1, 2 and 4, swaps 3 and 5, swaps 7 and 6.
Determining orbits and stabilizers

- \( \tau^* = (B_1, B_2, B_3, B_4, B_5, B_6, B_7) \in G \), so \( |\text{Orb}_G(B_1)| = 7 \).

- \( \rho^* = (B_1)(B_2, B_3, B_5)(B_4, B_7, B_6) \in H = \text{Stab}_G(B_1) \), and \( \rho = (1, 2, 4)(3, 6, 5)(7) \), so \( |\text{Orb}_H(1)| = 3 \).

- \( \sigma = (1)(2, 4)(3, 5, 7, 6) \) fixes 1 and swaps 2 and 4, so \( \sigma^* = (B_1)(B_2, B_4, B_6, B_3)(B_5, B_7) \in K = \text{Stab}_H(1) \), and \( |\text{Orb}_K(2)| = 2 \).

- \( \delta = (1)(2)(4)(3, 5)(7, 6) \) fixes 1, 2 and 4, so \( \delta^* = (B_1)(B_2)(B_3, B_4)(B_5, B_7)(B_6) \in L = \text{Stab}_K(2) \). In fact, \( L = \{id, \delta^*, \sigma^* \delta^*, \delta^* \sigma^* \delta^* \} \) and \( |L| = 4 \).
Let’s “do the math”

The Orbit-Stabilizer Theorem tells us that for \( G = \text{Sym}(D) \),

\[
|G| = |\text{Orb}_G| \cdot |\text{Orb}_H(1)| \cdot |\text{Orb}_K(2)| \cdot |\text{Stab}_K(2)|
= 7 \cdot 3 \cdot 2 \cdot 4
= 168.
\]

Hence, there are 168 symmetries of \((7, 3, 1)\).
Facts about $\text{Sym}((7,3,1))$

- $\text{Sym}((7,3,1))$ is generated by $\tau = (1,2,3,4,5,6,7)$ and $\sigma = (1)(2,4)(3,5,7,6)$.

- $\text{Sym}((7,3,1))$ is commonly known as $\text{GL}(3,2)$, the $3 \times 3$ matrices with entries in $\mathbb{Z}$ mod 2.

- Another name for $\text{GL}(3,2)$ is $\text{PSL}(2,7)$, the $2 \times 2$ matrices with entries in $\mathbb{Z}$ mod 7 and determinant 1, with $I$ and $-I$ identified.

- $\text{Sym}((7,3,1))$ is simple: it has no nontrivial normal subgroups.

- ... and ...
The Surprise Ending

Brown

The symmetries of (7, 3, 1)
Sym((7,3,1)) contains within its subgroup structure a copy of the (7, 3, 1) design.
THANK YOU!