What is a vector in hyperbolic geometry? And, what is a hyperbolic linear transformation?

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1 Introduction

Many mathematics undergraduate students are familiar with vectors in the Euclidean plane. For instance, these students understand that a vector is uniquely determined by its length and direction. Also, these students know that the sum of two given vectors is a vector that could be identified with a diagonal of the parallelogram whose adjacent sides are the two given vectors. The students also know that if one multiplies a real number to a vector then the result is a vector whose length is the product of the length of the original vector and the absolute value of the real number, and the direction of the resulting vector is either the same direction or the opposite direction of the original vector. Moreover, those students who have studied linear algebra would know that linear transformations are functions that are defined on vectors and whose range also consist of vectors, and these linear transformations *preserve* addition of vectors and the multiplication of a real number to a vector. A linear transformation, for example, could rotate each vector through a specific angle about its initial point.

In this paper, we will introduce students to the study of vectors in hyperbolic geometry. We show that a **hyperbolic vector** is uniquely defined by its **hyperbolic length** and **hyperbolic direction**. Also, we show that two hyperbolic vectors could be added, that one could multiply a real number to a hyperbolic vector, and there are **hyperbolic linear transformations** which preserve the addition of two hyperbolic vectors and the multiplication of a real number to a hyperbolic vector. It should not come as a surprise that there would be striking differences between the study of vectors in Euclidean geometry and hyperbolic geometry. For instance, if a hyperbolic linear transformation is nonzero then it is necessarily bijective.

The outline of this paper is described as follows. In Section 2, we discuss directed segments of hyperbolic lines and hyperbolic vectors. In Section 3, we discuss hyperbolic linear transformations.

The results in this paper are well-known, and we intend to present hyperbolic geometry to both students and teachers in an elementary fashion. Appropriate prerequisites for this article are calculus and linear algebra. A good introduction to hyperbolic geometry is [2], and an elementary analysis of the hyperbolic lines or so-called geodesics in hyperbolic geometry can be found in [3].

2 Hyperbolic Lines and Hyperbolic Vectors

In this section, we introduce the idea of a hyperbolic vector. Let $\langle\ ,\ \rangle$ be the usual inner-product on the set $\mathbb{C}$ of complex numbers defined by

$$\langle z, w\rangle = \mathrm{Re}(z\overline{w}) \text{ where } z, w \in \mathbb{C}.$$

A standard model used in the study of hyperbolic geometry is the Poincare disk

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

see [2, 3]. Another commonly used model is the upper-half plane $\{x + iy \in \mathbb{C} : y > 0\}$ which is also discussed in [2]. For the remainder of this article, we restrict ourselves to the Poincare disk model.

We call a function $\gamma(t) = x(t) + i \cdot y(t) \in D$ with $t \in [a, b]$ a curve segment in D if $x(t)$ and $y(t)$ are real-valued analytic functions of t defined on a domain that contains the closed interval $[a, b]$, and the derivatives $x'(t)$ and $y'(t)$ are not simultaneously zero for each t. The **hyperbolic length** of γ is defined to be

$$L(\gamma) = \int_a^b \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)} dt. \tag{2.1}$$

A curve segment joining two points q and p (i.e., $\gamma(a) = q$ and $\gamma(b) = p$) which minimizes (2.1) among all curve segments joining q to p is called a **hyperbolic line** or **geodesic**, and the corresponding minimum value of (2.1) is the **hyperbolic distance** between q and p. (See Lemma 9.8, p. 53 in [1]). The hyperbolic lines are open diameters in D and open arcs of circles perpendicular to the boundary of D (see p. 94 in [1] or p. 80 in [2]). Moreover, the hyperbolic distance between q and p is (see p. 95 in [2])

$$d(q,p) = \ln\left(\frac{|1-\overline{q}p| + |p-q|}{|1-\overline{q}p| - |p-q|}\right). \tag{2.2}$$

Next, we use the binary operations $\oplus$ and $\odot$ on D which are the analogues of vector addition and multiplication of a real number to a vector in the Euclidean plane. These binary operations are discussed in-depth in [3]. That is, for $x, z_1, z_2 \in D$ and $t \in \mathbb{R}$ we have

$$z_1 \oplus z_2 = \frac{z_1 + z_2}{1 + \overline{z_1} z_2} \qquad \text{and} \qquad t \odot x = \frac{x}{|x|}\frac{(1+|x|)^t - (1-|x|)^t}{(1+|x|)^t + (1-|x|)^t}. \tag{2.3}$$

Remark 1. The proofs of the following four identities are left as exercises for the reader to verify. The proofs follow from (2.2) and (2.3).

(a) $d\left(\frac{e^{d(q,p)}-1}{e^{d(q,p)}+1} \cdot \frac{(-q)\oplus p}{|(-q)\oplus p|}, 0\right) = d(q,p)$

(b) $|(-q)\oplus p| = \frac{e^{d(q,p)}-1}{e^{d(q,p)}+1}$

(c) $\frac{(-v)\oplus(v\oplus w)}{|(-v)\oplus(v\oplus w)|} = \frac{w}{|w|}$

(d) $d(v\oplus w, v) = d(w, 0)$ □

In Euclidean geometry, many students have seen that the graph of the parametric equation

$$l(t) = (x_0, y_0) + t(-x_0 + x_1, -y_0 + y_1)$$

with parameter t is a line through the points (x_0, y_0) and (x_1, y_1) where $l(0) = (x_0, y_0)$ and $l(1) = (x_1, y_1)$. Similarly, a hyperbolic line can be parametrized as the next theorem shows.

Theorem 1 Parametrizing a Hyperbolic Line

Let $q \neq p$. A parametric equation of the hyperbolic line through q and p is

$$h_{qp}(t) = q \oplus (t \odot ((-q)\oplus p)) \;\text{ with } t \in \mathbb{R}. \tag{2.4}$$

Furthermore, h_{qp} satisfies

$$\text{(a)}\;\; h_{qp}(0) = q, h_{qp}(1) = p, \quad \text{and} \quad \text{(b)}\;\; \frac{h'(0)}{|h'(0)|} = \frac{(-q)\oplus p}{|(-q)\oplus p|}.$$

Proof The proof of (2.4) is discussed in detail in [3]. To prove part (b), we note that

$$\left.\frac{d}{dt}\right|_{t=0} (t\odot x) = \frac{x}{|x|}\left(2(\ln(1+|x|) - \ln(1-|x|)\right) = 2d(0,x)\frac{x}{|x|} \tag{2.5}$$

and

$$\frac{d}{dt}\left(x \oplus t\right) = \frac{1-|x|^2}{\left(1+\overline{x}t\right)^2}.$$

Moreover, if $t \mapsto f(t) \in D$ is a differentiable function then by the chain rule we obtain

$$\frac{d}{dt}(x \oplus f(t)) = \frac{1-|x|^2}{\left(1+\overline{x}f(t)\right)^2} f'(t). \tag{2.6}$$

Therefore, by combining (2.5) and (2.6) we obtain

$$h'_{qp}(0) = \left.\frac{d}{dt}\right|_{t=0} \left(q \oplus \left(t \odot ((-q) \oplus p)\right)\right) = 2d(0,(-q)\oplus p) \cdot \frac{1-|q|^2}{|(-q)\oplus p|} \cdot (-q) \oplus p$$

and part (b) of the theorem follows immediately. □

In the introduction, we implied that two vectors in the Euclidean plane are the same if the directions and lengths of the vectors are the same, respectively. We will also use the geometric quantities of 'direction' and 'length' to help us intuitively understand the idea of a hyperbolic vector.

If h is a parametrization of a hyperbolic line, we call the set $\{h(t) : 0 \le t \le 1\}$ a **directed hyperbolic segment** with initial point $h(0)$, terminal point $h(1)$, with hyperbolic length $d_1 = d(h(0), h(1))$, and hyperbolic direction $z = h'(0)/|h'(0)|$. (The quantity z corresponds to the slope of a line in the Euclidean plane since z is the unit tangent vector to the curve segment h). We now define two directed hyperbolic segments to be **equivalent** if their hyperbolic lengths and hyperbolic directions are the same.

Definition 1 Hyperbolic Vector

A hyperbolic vector is the equivalence class of a directed hyperbolic segment.

We denote the equivalence class of the hyperbolic line segment

$$\{h_{qp}(t) : 0 \le t \le 1\}$$

with initial point q and terminal point p by $\overrightarrow{qp}$.

When two hyperbolic vectors $\overrightarrow{qp}$ and $\overrightarrow{rs}$ represent the same equivalence classes, we say the hyperbolic vectors are equal and we write $\overrightarrow{qp} = \overrightarrow{rs}$. Using Definition 1 and Theorem 1(b) we see that $\overrightarrow{qp} = \overrightarrow{rs}$ if and only if

$$\frac{(-q) \oplus p}{|(-q) \oplus p|} = \frac{(-r) \oplus s}{|(-r) \oplus s|} \text{ and } d(q,p) = d(r,s).$$

Moreover, by using the identity in Remark 1(a) we obtain $\overrightarrow{qp} = \overrightarrow{rs}$ if and only if

$$\frac{e^{d(q,p)} - 1}{e^{d(q,p)} + 1} \cdot \frac{(-q) \oplus p}{|(-q) \oplus p|} = \frac{e^{d(r,s)} - 1}{e^{d(r,s)} + 1} \cdot \frac{(-r) \oplus s}{|(-r) \oplus s|}.$$

Thus, the mapping

$$\overrightarrow{qp} \mapsto \frac{e^{d(q,p)} - 1}{e^{d(q,p)} + 1} \cdot \frac{(-q) \oplus p}{|(-q) \oplus p|} \tag{2.7}$$

is a bijection from the set of all hyperbolic vectors onto the Poincare disk D. Therefore, we could identify the points in D with the set of all hyperbolic vectors.

The mapping (2.7) can be given a geometric interpretation. To do this, note that the identity in Remark 1(b) could be used to rewrite (2.7) as the mapping

$$\overrightarrow{qp} \mapsto (-q) \oplus p. \tag{2.8}$$

Since the vector in the Euclidean plane joining the point (x_0, y_0) to (x_1, y_1) is usually denoted by $\langle -x_0 + x_1, -y_0 + y_1 \rangle$, analogously the mapping (2.8) implies that the hyperbolic vector joining q to p can be identified with the point $(-q) \oplus p$ in D.

To motivate the geometric meaning of the next theorem, we recall that the initial point of a vector in the Euclidean plane can be chosen to be any point in the plane (the terminal point of course would depend on the initial point). The same is true in hyperbolic geometry, that is, given a hyperbolic vector and a point v in D there is is an equivalent hyperbolic vector whose initial point is v.

Theorem 2 Changing the Initial Point of a Hyperbolic Vector
Let $v, w \in D$. The hyperbolic vector $\overrightarrow{0w}$ is equivalent to the hyperbolic vector with initial point v and terminal point $v \oplus w$.

Proof Theorem 1 shows that the parametric equation

$$l(t) = v \oplus (t \odot ((-v) \oplus (v \oplus w)))$$

with $t \in I\!R$ satisfies $l(0) = v$, $l(1) = v \oplus w$, and

$$\frac{l'(0)}{|l'(0)|} = \frac{(-v) \oplus (v \oplus w)}{|(-v) \oplus (v \oplus w)|}.$$

By Remarks 1(c) and 1(d), we obtain

$$\frac{l'(0)}{|l'(0)|} = \frac{w}{|w|} \text{ and } d(v \oplus w, v) = d(w, 0),$$

respectively. Hence, by Definition 1 the hyperbolic vector $\overrightarrow{0w}$ is equivalent or equal to the vector with intial point v and terminal point $v \oplus w$. $\square$

At this point, we digress and recall a certain trigonometric hyperbolic formula. Three points A, B, and C in the Poincare disk D that are 'noncollinear' (i.e., the points do not lie in a single hyperbolic line) determine a unique **hyperbolic triangle** $\triangle ABC$. In addition, if two sides of $\triangle ABC$ are perpendicular (i.e., the tangent lines to the hyperbolic lines at a vertex, say at C, are perpendicular), then we say that the hyperbolic triangle is a **hyperbolic right triangle**. In this case, we have the known formula

$$\sin \angle CAB = \frac{\sinh a}{\sinh c} \tag{2.9}$$

where $\angle CAB$ is the angle at A, and a and c are the hyperbolic lengths of the sides opposite vertices A and C, respectively.

In the next theorem, $\mathrm{Im}(w)$ denote the imaginary part of a complex number w.

Theorem 3 Angles of Hyperbolic Lines

Let $w \in D$ with $\mathrm{Im}(w) > 0$, and let $0 < \theta < \pi$. Then there exists a hyperbolic line with parametric equation l such that $l(0)$ is a point on the x-axis, $l'(0)/|l'(0)| = e^{i\theta}$, and $l(1) = w$

Proof Consider the three points in D given by

$$a = \frac{e-1}{e+1}, \; a_s = s \odot a, \text{ and } w_s = a_s \oplus w$$

with $s \in \mathbb{R}$. By applying Theorem 2, we obtain $\overrightarrow{0w} = \overrightarrow{a_s w_s}$.

Let a_s and w_s be the initial point and terminal point of $\overrightarrow{a_s w_s}$, respectively. By applying (2.9) to the hyperbolic right triangle with vertices a_s, w_s, and the point on the x-axis closest to w_s, one obtains that the perpendicular distances from w_s to the x-axis are equal for all s, as illustrated in Figure 1. Thus, the set of all terminal points $\{w_s : s \in \mathbb{R}\}$ is a curve that is equidistant from the x-axis.

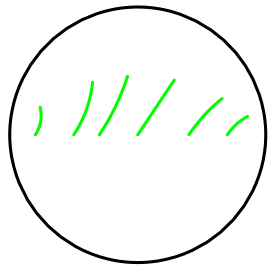

Figure 1. The terminal points of the vectors $\overrightarrow{a_s w_s}$ are equidistant from the x-axis.

Choose s_0 such that the angle joining the origin to w_{s_0} makes an angle of θ with the positive x-axis. In particular, we have

$$\frac{w_{s_0}}{|w_{s_0}|} = e^{i\theta}.$$

Then the hyperbolic line given by

$$l(t) = (-(a_{s_0})) \oplus (t \odot w_{s_0})$$

satisfies the conclusion of the Theorem. □

3 Hyperbolic Linear Transformations

Definition 2 Hyperbolic Linear Transformation

A mapping $L : D \to D$ is called a hyperbolic linear transformation if

$$L(v \oplus w) = L(v) \oplus L(w) \text{ and } L(t \odot v) = t \odot L(v)$$

for all $v, w \in D$ and $t \in I\!R$.

For example, the mappings $v \mapsto e^{i\alpha}v$ for $\alpha \in I\!R$, and $v \mapsto \overline{v}$ with $v \in D$ are hyperbolic linear transformations. Thus, hyperbolic linear transformations include rotations about the origin and reflections about lines passing through the origin.

The next two theorems and corollary show that a hyperbolic linear transformation is either bijective or is the zero mapping (the function which sends each vector to 0). This property of hyperbolic linear transformations is very different from linear transformations of the Euclidean plane.

Theorem 4 A Nonzero Hyperbolic Linear Transformation is Injective
If L is a hyperbolic linear transformation that is not injective, then L is the zero linear transformation.

Proof Let

$$a = \frac{e-1}{e+1} \qquad \text{and} \qquad b = \frac{e-1}{e+1} \cdot e^{i\pi/2}.$$

By applying a rotation, if needed, we could assume $L(a) = 0$. Let $w = \frac{e-1}{e+1} \cdot e^{i\alpha}$ where $0 < \alpha < \frac{\pi}{2}$. By forming the hyperbolic right triangle with vertices 0, w, and the point on the x-axis closest to w, we find that

$$w = (t_1 \odot a) \oplus (t_2 \odot b)$$

where

$$\sin\alpha = \frac{\sinh(t_2)}{\sinh(1)}$$

and α is the angle formed by the vector $\overrightarrow{0w}$ with the x-axis. By using Definition 2, we find

$$L(w) = t_2 \odot L(b) = \sinh^{-1}(\sinh(1)\sin(\alpha)) \odot L(b).$$

But

$$2 \odot L(w) = 2 \odot \left[\sinh^{-1}(\sinh(1)\sin(\alpha)) \odot L(b)\right] = \left(2\sinh^{-1}(\sinh(1)\sin(\alpha))\right) \odot L(b)$$

and

$$2 \odot L(w) = L(2 \odot w) = \sinh^{-1}(\sinh(2)\sin(\alpha)) \odot L(b).$$

Since

$$2\sinh^{-1}(\sinh(1)\sin(\alpha)) \neq \sinh^{-1}(\sinh(2)\sin(\alpha))$$

for $0 < \alpha < \frac{\pi}{2}$, it follows that $L(b) = 0$. Thus, $L = 0$ and the claim is proved. □

Theorem 5 An Injective Hyperbolic Linear Transformation is Surjective
If L is an injective hyperbolic linear transformation, then L is surjective.

Proof We use the points a and b defined in the proof of Theorem 4. By applying a rotation, if needed, we can assume $L(a)$ lies on the positive x-axis of D. Also, we could apply the reflection about the x-axis if needed, so that we can assume $\mathrm{Im}(L(b)) > 0$.

Let $0 < \theta < \pi$ be the angle the directed hyperbolic line segment joining the origin to $L(b)$ makes with the positive x-axis. Suppose $w \in D$ and $\mathrm{Im}(w) > 0$. By Theorem 3, there exists a hyperbolic line whose parametric equation l satisfies $l(0) = s_1 \odot L(a)$ for some $s_1 \in I\!R$, $l(1) = w$, and $\dfrac{l'(0)}{|l'(0)|} = e^{i\theta}$. Choose $t_1 \in I\!R$ such that

$$t_1 d(0, L(b)) = d(l(0), l(1)).$$

Then

$$w = (s_1 \odot L(a)) \oplus (t_1 \odot L(b)) = L((s_1 \odot a) \oplus (t_1 \odot b))$$

and consequently w belongs to the range of L.

On the other hand, if $\mathrm{Im}(w) < 0$, then $\mathrm{Im}(-w) > 0$ and $-w$ belongs to the range of L. So, $-w = L(v)$ for some $v \in D$. Since $w = -L(v) = L(-v)$, w belongs to the range of L. Hence, L is surjective. □

Finally, by applying Theorems 4 and 5 we obtain the next corollary.

Corollary 6 A Nonzero Hyperbolic Linear Transformation is Bijective
If L is a nonzero hyperbolic linear transformation, then L is bijective.

References

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