Optimal Investment Strategies: A Constrained Optimization Approach

Janet L. Waldrop
Mississippi State University
jlc3@ra.msstate.edu

Faculty Advisor: Michael Pearson
Pearson@math.msstate.edu
1 Introduction.

The ultimate goal of modern investment theory is to construct an optimal portfolio of investments, where a portfolio is simply a collection of risky investments. The Capital Asset Pricing Model (CAPM), developed in 1964 by William F. Sharpe, seeks to find the relationship between the way assets are priced and their risk. Sharpe’s work was an extension of the work done by Harry Markowitz, who developed a method of efficient portfolio selection. To understand both the CAPM and Markowitz’s portfolio selection technique, we first need to develop the method of Lagrange Multipliers.

2 Lagrange Multipliers.

Consider the problem of maximizing or minimizing a function subject to given constraints. One way to solve this optimization problem is the method of Lagrange Multipliers. Suppose we have a function $g$, and we want to maximize (or minimize) this function subject to the constraint $F(x) = c$ where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for $F = (f_1, f_2, ..., f_m)$. We assume that there exists an open subset $U$ of $\mathbb{R}^n$ such that $f_i$ is differentiable on $U$ for $i = 1, 2, ..., m$ and that all the points that satisfy the constraint function lie in $U$. Also, we assume that $g$ is differentiable on $U$. Recall that the Jacobian of $F$, denoted $JF(x)$, is given by

\[
 JF(x) = \begin{bmatrix}
 \nabla f_1(x) \\
 \vdots \\
 \nabla f_m(x)
 \end{bmatrix} = \begin{bmatrix}
 \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
 \vdots & \ddots & \vdots \\
 \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
 \end{bmatrix}.
\]

Note that $JF$ is a $m \times n$ matrix, so $JF(x)$ is a linear map from $\mathbb{R}^n \rightarrow \mathbb{R}^m$. If the rank of $JF = m \leq n$ on $F^{-1}(c)$, we say that $F$ has full rank on $F^{-1}(c)$. In other words, $\{\nabla f_1(x), ..., \nabla f_m(x)\}$ is a set of linearly independent vectors for any $x$ in $F^{-1}(c)$.

By the Implicit Function Theorem, there exists $\varphi : \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$ so that if $(x_1, ..., x_n)$ is near a local extremum $\overrightarrow{p} = (p_1, ..., p_n)$ then $(x_1, ..., x_n) = (x_1, ..., x_{n-m}, \varphi(x_1, ..., x_{n-m})) \in F^{-1}(c)$. Moreover, $\overrightarrow{p} = (p_1, ..., p_{n-m}, \varphi(p_1, ..., p_{n-m}))$. Furthermore, if $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is constrained by $F(x) = c$, then we are really just maximizing $G(x_1, ..., x_{n-m}) = g(x_1, ..., x_{n-m}, \varphi(x_1, ..., x_{n-m}))$. Now if $\overrightarrow{p} = (p_1, ..., p_{n-m}, p_{n-m+1}, ..., p_n) = (p_1, ..., p_{n-m}, \varphi(p_1, ..., p_{n-m}))$ is a maximum of $G$, then $G(p_1, ..., p_{n-m}) = g(p_1, ..., p_{n-m}, \varphi(p_1, ..., p_{n-m}))$ and
we need $\frac{\partial G}{\partial x_i}(p_1, \ldots, p_{n-m}) = 0$ for $i = 1, 2, \ldots, n-m$. Note that

$$\frac{\partial G}{\partial x_k} = \frac{\partial g}{\partial x_k} + \sum_{i=n-m+1}^{n} \left( \frac{\partial g}{\partial x_i} \frac{\partial x_i}{\partial x_k} \right), \quad k = 1, \ldots, n-m.$$ 

Since

$$
\begin{pmatrix}
\varphi_{n-m+1}(x_1, \ldots, x_{n-m}) \\
\vdots \\
\varphi_n(x_1, \ldots, x_{n-m})
\end{pmatrix}
= 
\begin{pmatrix}
x_{n-m+1} \\
\vdots \\
x_n
\end{pmatrix},
\frac{\partial x_i}{\partial x_k} = \frac{\partial \varphi_i}{\partial x_k}, \quad i = n-m+1, \ldots, n.
$$

Hence, $\vec{p}$ must be a solution to the system

$$\frac{\partial G}{\partial x_k} = \frac{\partial g}{\partial x_k} + \sum_{i=n-m+1}^{n} \left( \frac{\partial g}{\partial x_i} \frac{\partial \varphi_i}{\partial x_k} \right) = 0, \quad k = 1, \ldots, n-m.$$ 

In matrix form we can write this system as

$$
\begin{pmatrix}
\frac{\partial g}{\partial x_i} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\
\frac{\partial \varphi_{n-m+1}}{\partial x_1} & \cdots & \frac{\partial \varphi_{n-m+1}}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial \varphi_n}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_n}
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

If we let $A$ represent the above matrix, we have $\nabla g \cdot A = (0, \ldots, 0)_{1 \times n-m}$. Furthermore, our constraint function,

$$F(x_1, \ldots, x_{n-m}, \varphi(x_1, \ldots, x_{n-m})) = \vec{c}$$

gives

$$f_i(x_1, \ldots, x_{n-m}, \varphi(x_1, \ldots, x_{n-m})) = c_i \quad \text{for } i = 1, 2, \ldots, n-m,$$

which implies that

$$\frac{\partial f_i}{\partial x_i} + \sum_{i=n-m+1}^{n} \frac{\partial f_i}{\partial x_i} \frac{\partial \varphi_i}{\partial x_i} = 0 \text{ for } i = 1, 2, \ldots, n-m.$$ 

So, $\nabla f_i \cdot A = (0, \ldots, 0)_{1 \times n-m}$ as well. Now, $A$ is a linear map from $\mathbb{R}^{n-m} \to \mathbb{R}^n$, so the rank of $A \leq n-m$. Furthermore, the upper portion of $A$ forms the $(n-m) \times (n-m)$ identity matrix, and $A$ has at least $n-m$ linearly independent rows. Thus, the rank of $A$ must be at least $(n-m)$ and is therefore equal to $(n-m)$. But, $\nabla g(\vec{p}), \nabla f_1(\vec{p}), \ldots, \nabla f_m(\vec{p})$ are all solutions to the homogeneous system

$$A^T \begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m \times 1}.$$ 

As we have already seen, the rank $A = (n-m)$, so the dimension of the kernel of $A$ is $n - (n-m) = m$. Furthermore, since we have $m+1$ solutions, the set must be linearly dependent. But we assumed that
the set \( \{ \nabla f_1(\vec{x}), \ldots, \nabla f_m(\vec{x}) \} \) is linearly independent, so \( \nabla g \) is in the span of \( \{ \nabla f_1(\vec{x}), \ldots, \nabla f_m(\vec{x}) \} \). In other words, there exist \( m \) scalars \( \lambda_1, \lambda_2, \ldots, \lambda_m \) such that \( \nabla g(\vec{p}) = \lambda_1 \nabla f_1(\vec{p}) + \ldots + \lambda_m \nabla f_m(\vec{p}) \). We call the scalars \( \lambda_1, \lambda_2, \ldots, \lambda_m \) Lagrange multipliers. So this gives us \( n \) equations, and the constraint \( F(\vec{x}) = \vec{c} \) gives us \( m \) equations for a total of \( n + m \) (possibly nonlinear) equations with \( n + m \) unknowns, namely \( \{ x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m \} \). Solving this system of equations is referred to as the method of Lagrange Multipliers. The solutions to these equations supply the possible solutions to the constrained optimization problem.

## 3 Efficient Frontier.

We will now use the method of Lagrange Multipliers to construct what is referred to as the efficient frontier of investment portfolios. That is, the portfolios that have the least amount of risk for a given level of return. Note that a portfolio is simply a collection of risky investments. The underlying assumptions of the model are homogeneity of investor expectations, that is all investors agree on expected values, standard deviations, and covariances, and all investors are able to borrow and lend on equal terms at some risk free rate. We know that the risk of a particular security can be measured by the variance or standard deviation of that security from its expected return. Similarly, the risk of an investment portfolio can be measured by the standard deviation or variance of that portfolio from its expected return. So our problem becomes that of minimizing portfolio risk for a given level of return.

Consider a portfolio consisting of \( n \) risky investments, and let \( w_i \) represent the proportion of the total investment placed in a particular investment \( i \). Also, let \( E(r_i) \) denote the expected return of \( i \) and \( \sigma_p^2 \) be the variance of the portfolio. The return of a portfolio is given by \( \sum w_i E(r_i) \), and it can be shown that the variance is given by \( \sum w_i w_j \text{cov}(i, j) \), where \( \text{cov}(i, j) \) is the covariance between returns from investments \( i \) and \( j \). Thus, our problem is to minimize

\[
\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(i, j)
\]

subject to the constraints

\[
\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} w_i E(r_i) = \mu,
\]

where \( \mu \) is the desired return of the portfolio. In this case, we can let

\[
f_1(w_1, \ldots, w_n) = w_1 + \ldots + w_n \quad \text{and} \quad f_2(w_1, \ldots, w_n) = E(r_1)w_1 + \ldots + E(r_n)w_n.
\]

Then

\[
\nabla f_1 = \left( \frac{\partial f_1}{\partial w_1}, \ldots, \frac{\partial f_1}{\partial w_n} \right) = (1, \ldots, 1) \quad \text{and} \quad \nabla f_2 = \left( \frac{\partial f_2}{\partial w_1}, \ldots, \frac{\partial f_2}{\partial w_n} \right) = (E(r_1), \ldots, E(r_n)),
\]

and using the method of Lagrange Multipliers, we seek \( \lambda_1 \) and \( \lambda_2 \) such that \( \nabla \sigma_p^2 = \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2 \). In order to determine \( \nabla \sigma_p^2 \), consider

\[
\frac{\partial \sigma_p^2}{\partial w_i} = \frac{\partial}{\partial w_i} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(i, j) \right) = 2w_i \text{cov}(i, i) + 2 \sum_{j=1, j \neq i}^{n} w_j \text{cov}(i, j) = 2 \sum_{j=1}^{n} w_j \text{cov}(i, j).
\]

Thus,

\[
\nabla \sigma_p^2 = \begin{pmatrix}
2w_1 \text{cov}(1, 1) + \ldots + 2w_n \text{cov}(1, n) \\
2w_1 \text{cov}(2, 1) + \ldots + 2w_n \text{cov}(2, n) \\
\vdots \\
2w_1 \text{cov}(n, 1) + \ldots + 2w_n \text{cov}(n, n)
\end{pmatrix} = 2 \begin{pmatrix}
\text{cov}(1, 1) & \cdots & \text{cov}(1, n) \\
\vdots & \ddots & \vdots \\
\text{cov}(1, n) & \cdots & \text{cov}(n, n)
\end{pmatrix} \begin{pmatrix}
w_1 \\
\vdots \\
w_n
\end{pmatrix}
\]
Since $\text{cov}(i, j) = \text{cov}(j, i)$, by the Method of Lagrange Multipliers, we seek to solve
\[
\nabla \sigma_p^2 = \lambda_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} E(r_1) \\ \vdots \\ E(r_n) \end{pmatrix} = \begin{pmatrix} 1 & E(r_1) \\ \vdots & \vdots \\ 1 & E(r_n) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.
\]
If we absorb the 2 into our constants $\lambda_1$ and $\lambda_2$, we obtain
\[
\begin{pmatrix} \text{cov}(1, 1) & \cdots & \text{cov}(1, n) \\ \vdots & \ddots & \vdots \\ \text{cov}(1, n) & \cdots & \text{cov}(n, n) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 1 & E(r_1) \\ \vdots & \vdots \\ 1 & E(r_n) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.
\]
Hence, our system of equations consists of the two constraint functions
\[
\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} E(r_i) w_i = \mu,
\]
and the following equations constructed using Lagrange Multipliers:
\[
\sum_{i=1}^{n} w_i \text{cov}(1, i) = \lambda_1 + \lambda_2 E(r_1)
\]
\[
\vdots
\]
\[
\sum_{i=1}^{n} w_i \text{cov}(n, i) = \lambda_1 + \lambda_2 E(r_n)
\]
This system can be written in matrix form as follows:
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 \\
E(r_1) & E(r_2) & \cdots & E(r_n) & 0 & 0 \\
\text{cov}(1, 1) & \text{cov}(1, 2) & \cdots & \text{cov}(1, n) & -1 & -E(r_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\text{cov}(1, n) & \text{cov}(2, n) & \cdots & \text{cov}(n, n) & -1 & -E(r_2)
\end{pmatrix}
\begin{pmatrix}
w_1 \\
\vdots \\
w_n
\end{pmatrix}
= \begin{pmatrix}
1 \\
\mu \\
\vdots \\
\lambda_1 \\
\lambda_2
\end{pmatrix}.
\]
Now let the symmetric matrix of covariances be denoted by $V$, let $R = \begin{pmatrix} 1 & \cdots & 1 \\ E(r_1) & \cdots & E(r_n) \end{pmatrix}$, and let $\overrightarrow{w}$ and $\overrightarrow{\lambda}$ represent the $n \times 1$ weight vector and $2 \times 1$ Lagrange multiplier vector respectively. Then our system in (2) can be represented by the block matrices
\[
\begin{pmatrix}
R \\
V
\end{pmatrix}
\begin{pmatrix}
\overrightarrow{w} \\
\overrightarrow{\lambda}
\end{pmatrix}
= \begin{pmatrix}
1 \\
\mu
\end{pmatrix}.
\]
and our system under the method of Lagrange Multipliers in (1) can be represented as
\[
V \overrightarrow{w} = R^T \overrightarrow{\lambda} \quad \text{or} \quad \overrightarrow{w} = V^{-1} R^T \overrightarrow{\lambda}.
\]
Recall that $\sigma_p^2 := \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(i, j)$, which can be represented in matrix form as
\[
\sigma_p^2 = \overrightarrow{w}^T V \overrightarrow{w}.
\]
Also, note that in order for $V^{-1}$ to exist, the rows of $V$ must be linearly independent, which is equivalent to the assumption that no single investment in our portfolio can be replaced with a linear combination of
other investments. We can make this assumption, since if our portfolio contained an investment that was
dependent we could replace it with the appropriate linear combination of investments without changing the
characteristics of our portfolio. Therefore, we assume that the rows of $V$ are linearly independent, and thus
$V^{-1}$ exists. Now, substituting (4) into (3), we obtain

$$
\begin{pmatrix}
R & 0_{2 \times 2} \\
V & -R^T
\end{pmatrix}
\begin{pmatrix}
V^{-1}R^T \bar{\lambda} \\
\bar{\lambda}
\end{pmatrix}
= \begin{pmatrix}
1 \\
\mu \\
0_{n \times 1}
\end{pmatrix}
$$

$$
\implies \begin{pmatrix}
RV^{-1}R^T \bar{\lambda} \\
VV^{-1}R^T \bar{\lambda} - R^T \bar{\lambda}
\end{pmatrix}
= \begin{pmatrix}
1 \\
\mu \\
0_{n \times 1}
\end{pmatrix}
$$

$$
\implies \begin{pmatrix}
RV^{-1}R^T \bar{\lambda} \\
0_{n \times 1}
\end{pmatrix}
= \begin{pmatrix}
1 \\
\mu \\
0_{n \times 1}
\end{pmatrix}
$$

$$
\implies RV^{-1}R^T \bar{\lambda} = \begin{pmatrix}
1 \\
\mu
\end{pmatrix}
$$

Letting $H = RV^{-1}R^T$,

$$
H \bar{\lambda} = \begin{pmatrix}
1 \\
\mu
\end{pmatrix}
\quad \text{and} \quad \bar{\lambda} = H^{-1} \begin{pmatrix}
1 \\
\mu
\end{pmatrix}
$$

(5)

Note that $R$ is $2 \times n$, $V^{-1}$ is $n \times n$, and $R^T$ is $n \times 2$, so $H = RV^{-1}R^T$ is $2 \times 2$. Furthermore, $V^{-1}$ is
symmetric, so $H = RV^{-1}R^T = (RV^{-1}R^T)^T = H^T$ and $H$ is symmetric.

Using (4), we get

$$
\sigma_p^2 = \bar{w}^T V V^{-1} R^T \bar{\lambda}
$$

$$
= \bar{w}^T R^T \bar{\lambda},
$$

and (5) gives

$$
\sigma_p^2 = \bar{w}^T R^T H^{-1} \begin{pmatrix}
1 \\
\mu
\end{pmatrix}
$$

$$
= (R \bar{w})^T H^{-1} \begin{pmatrix}
1 \\
\mu
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & \mu
\end{pmatrix} H^{-1} \begin{pmatrix}
1 \\
\mu
\end{pmatrix},
$$

(6)

since $R \bar{w} = \begin{pmatrix}
1 \\
\mu
\end{pmatrix}$ is one of the constraints.

We have already noted that $H$ is a $2 \times 2$ symmetric matrix, so define $H$ as

$$
H = \begin{pmatrix}
a & b \\
b & c
\end{pmatrix}
$$

for some $a, b, c \in \mathbb{R}$

Then we can compute $H^{-1}$, namely

$$
H^{-1} = \frac{1}{ac - b^2} \begin{pmatrix}
c & -b \\
-b & a
\end{pmatrix}
$$
and equation (6) becomes

\[
\sigma_p^2 = (1 \quad \mu) H^{-1} \begin{pmatrix} 1 \\ \mu \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} 1 \\ \mu \end{pmatrix} \begin{pmatrix} c - b \mu \\ -b + a \mu \end{pmatrix} \\
= \frac{a \mu^2 - 2b \mu + c}{ac - b^2} = \alpha \mu^2 + \beta \mu + \gamma.
\]

Hence, our variance is a quadratic function of the expected return \( \mu \). If we plot the return on the \( x \)-axis and the risk, or variance, on the \( y \)-axis, the right half of the parabola determined by this function defines our efficient frontier. We only need to consider the right half of the parabola, as any point on the left half gives a lower return for the same amount of risk.

4 Example.

Using the daily stock prices of several stocks, we can actually construct the efficient frontier for portfolios consisting of these stocks. In order to calculate the needed statistics, the daily stock prices were recorded from July 3, 1962 to December 31, 1998 for the following stocks: S&P 500, Coca-Cola, Disney, IBM, International Paper, and Philip Morris (See Appendix). From this data, the return of each individual stock, \( r_t \), was calculated for each day so that the average daily return given by

\[
\eta = \frac{\sum_{t=1}^{N} r_t}{N}, \quad \text{where } N \text{ is the number of observations},
\]

could be calculated. Using this data, the covariances between each pair of stocks was then calculated by

\[
Cov(r_A, r_B) = \frac{\sum_{t=1}^{N} (r_{A,t} - \eta_A)(r_{B,t} - \eta_B)}{N - 1}.
\]

Using the construction in the previous section, we find for this particular example that the coefficients for the polynomial defining our efficient frontier are

\[
\alpha = 2.53153 \times 10^6 \\
\beta = -1.58786 \times 10^3 \\
\gamma = 1.00042
\]
From this data, we can generate the following plot of the efficient frontier:

![Efficient Frontier Graph]

5 Optimal Investment Strategy.

In order to determine the optimal investment for a particular investor, we must first introduce the idea of utility functions. A utility function for a particular investor measures the effect of the investment or portfolio on the investor's level of wealth. For the CAPM, we assume that investors utility functions depend only on an investment’s risk and expected return, that is \( U = f(E(r), \sigma^2) \). As investors prefer a higher expected return versus a lower return, we can also assume that \( \frac{\partial U}{\partial E(r)} > 0 \). Furthermore, we assume that investors are risk averse, that is given an expected return, investors will choose the lowest possible risk. Mathematically, risk averse simply means that \( \frac{\partial^2 U}{\partial \sigma^2} < 0 \). The set of level curves to a particular utility function are called the indifference curves for that investor. Based on our above assumptions, we know that these indifference curves will be upward sloping and that each indifference curve represents a given level of expected utility. As we move down and to the right on the set of indifference curves, we reach a more desirable position.

Next, we introduce the idea of a risk free asset denoted \( r_f \), that is an asset with \( \sigma^2_{r_f} = 0 \) and expected return equal to that of the pure interest rate, say \( \mu_{r_f} \). Suppose we want to invest a portion of our wealth \( \alpha \) in the risk free asset and the rest in a portfolio \( A \). Then the expected return of this investment is given by

\[
E(r) = \alpha E(r_{r_f}) + (1 - \alpha)E(r_A) = \alpha \mu_{r_f} + (1 - \alpha)E(r_A)
\]

and

\[
\sigma^2 = \alpha^2 \sigma^2_{r_f} + (1 - \alpha)^2 \sigma^2_A + 2E(r_A)\sigma_A \sigma_{r_f} = (1 - \alpha)^2 \sigma^2_A \quad \Rightarrow \quad \sigma = (1 - \alpha)\sigma_A,
\]

a linear relationship. Thus, the variance of any investment combining some portfolio and a risk free asset must lie along the line passing through the points in the return-risk (or return-variance) plane representing the risk free asset and that portfolio.
We have already shown that the optimal investments lie along the efficient frontier, so the line that we are interested in is the line tangent to the efficient frontier and passing through the risk free asset. This line is referred to as the Capital Market Line. Using basic calculus techniques, we can easily construct this tangent line. The equation of our efficient frontier is a quadratic, namely \( \sigma^2_p = \alpha \mu^2 + \beta \mu + c \), so the slope of the tangent line at any point is \( 2\alpha \mu + \beta \). We also know that our tangent line must intersect the efficient frontier at some point of the form \((\hat{x}, a\hat{x}^2 + b\hat{x} + c)\) and that it must pass through the point \((r_f, 0)\). Now, we have the simple problem of finding the equation of a line with slope \(2\alpha \hat{x} + \beta\) and passing through the points \((\hat{x}, a\hat{x}^2 + b\hat{x} + c)\) and \((r_f, 0)\) where \(\hat{x}\) is our only unknown. So we have

\[
\begin{align*}
a\hat{x}^2 + b\hat{x} + c - 0 &= 2a\hat{x} + b(\hat{x} - r_f) \\
a\hat{x}^2 - 2ar_f\hat{x} - r_fb - c &= 0
\end{align*}
\]

and the equation of our line is one of the equations

\[
y = \left(2 \left( ar_f \pm \sqrt{a^2r_f^2 + abr_f + ac} \right) + b \right) (x - r_f).
\]

Of course, we would choose the line that is tangent to the right side of the parabola and ignore the tangent line on the left.

The optimal investment is then the investment along this line that is tangent to a particular investor’s indifference curves.

6 Example.

If we go back to our previous example, where we constructed the efficient frontier, we can also construct the line on which our optimal investment must lie. As stated previously, we know that this line must be tangent to the efficient frontier and must pass through the point on the \(x\)-axis representing the risk free rate. For our example, we will assume a risk free rate of 5% per year. As our stock returns are given in terms of daily returns, we must first convert this yearly rate to a daily rate:

\[
(1 + x)^{252} = 1.05 \\
\implies x = .019\%.
\]

Here is a plot of the line passing through the point \((.00019,0)\) and tangent to the efficient frontier together
7 The Capital Asset Pricing Model

William Sharpe, in his 1964 paper, extended the ideas presented thus far to argue that, under appropriate assumptions, the price of a particular asset would adjust in equilibrium so that the asset would plot along the capital market line. Thus, in equilibrium, the price and risk of an asset are related linearly. Recall that an underlying assumption of our model is the homogeneity of investor expectations, and therefore, the model does not capture all aspects of market behavior. However, modern investment theory, in particular the study of equilibrium pricing of assets under conditions of uncertainty, is based largely on the early work of Markowitz and Sharpe.

8 Bibliography

References


