Small Perimeter-Minimizing Double Bubbles in Compact Surfaces are Standard

Frank Morgan
Department of Mathematics and Statistics
Williams College
Williamstown, Massachusetts 01267
Frank.Morgan@williams.edu

Abstract

We prove that in a smooth, compact, two-dimensional submanifold of $\mathbb{R}^N$, the least-perimeter way to enclose and separate two regions of small prescribed areas is a standard double bubble, consisting of three constant-curvature curves meeting in threes at 120 degrees.

This paper is largely superseded by [MW], which proves that small stable double bubbles are standard.

Mathematics Subject Classification: 53A10, 49Q20
Key words and phrases: double bubble, soap bubble

I would like to thank my gracious hosts, especially Alan Paterson and Gerard Buskes, of the Louisiana/ Mississippi Section of the Mathematical Association of America 78th annual meeting at The University of Mississippi, March 23-24, 2001. Since the material of my invited address on the “Proof of the Double Bubble Conjecture” appears elsewhere ([M1—4], [HMRR]), I have submitted this paper instead. This research is partially supported by a National Science Foundation grant.
1. Introduction

Cotton and Freeman [CF, Conj. 1.1] conjectured that the least-perimeter way to enclose and separate two small prescribed volumes in a compact Riemannian manifold \( M \) is a standard double bubble (consisting of three constant-mean-curvature discs meeting along a curve at 120 degrees). Such a result was known only for \( \mathbb{R}^2 \) [F], \( S^2 \) [Ma], \( H^2 \) [CF, Thm. 2.4], \( \mathbb{R}^3 \) [HMRR], and \( \mathbb{R}^4 \) [RHLS], where it was known for arbitrary prescribed volumes or areas. Our Theorem 2.1 provides a proof for any compact, two-dimensional submanifold of \( \mathbb{R}^N \).

1.1. The proof. The proof is a limit argument: in the small a surface is nearly Euclidean. The difficulty is the need for curvature bounds to obtain smooth convergence. If the ratio of the smaller area to the larger is bounded away from 0, such curvature bounds may be obtained from a rescaled limit bubble. If not, the rescaled limit is a circle, but the bubbles near the limit may have a large number \( m_i \) of tiny components of the smaller region. The requisite bound on \( m_i \) follows from a decomposition argument and an application of the isoperimetric inequality.

1.2. Higher dimensions. Generalizing the proof to \( n \)-dimensional manifolds \( M \), even when the ratio of the smaller volume to the larger is bounded away from 0, would require knowing the result in \( \mathbb{R}^n \) and knowing that convergence weakly and in measure, under bounded mean curvature, implies \( C^1 \) convergence, as is obvious for curves and known for \( (n-1) \)-dimensional surfaces without singularities [A, Sect. 8]. For \( n > 2 \), the argument needs a volume concentration lemma such as [M2, 13.7(1)] (with the partitioning of \( \mathbb{R}^n \) into cubes replaced by a bounded-multiplicity covering of \( M \) by balls). Our approach does give an alternate proof of [M], Thm. 2.2] that small single bubbles are nearly round balls.

1.3. Existence and regularity. Geometric measure theory ([M4], cf. [M2, 13.4, 13.10]) proves that a perimeter-minimizing double bubble exists and consists of finitely many constant-curvature curves meeting in threes at 120 degrees. All curves separating the same two regions or exterior have the same curvature.
2. Small Double Bubbles are Standard

2.1. Theorem. Let \( M \) be a smooth, compact, two-dimensional submanifold of \( \mathbb{R}^N \). For small prescribed areas, a perimeter-minimizing double bubble is standard, i.e., consists of three constant-curvature curves meeting in two points at 120 degrees.

Proof. We claim that it suffices to show that a double bubble of small areas \( A \geq A' > 0 \), which minimizes perimeter among double bubbles of small areas at least \( A, A' \), is standard. Indeed, given small areas \( A_0, A'_0 \), minimize perimeter among double bubbles of small areas \( A \geq A_0, A' \geq A'_0 \). We may assume \( A \geq A' \). The bubble certainly minimizes perimeter among double bubbles of small areas at least \( A, A' \). By assumption it is standard, and therefore its curvatures \( \kappa_1, \kappa_2 \) are positive. Therefore it must have areas exactly \( A_0, A'_0 \), or by reducing area one could reduce perimeter. Therefore a perimeter minimizer among double bubbles of areas \( A_0, A'_0 \) is actually a perimeter minimizer among double bubbles of small areas at least \( A_0, A'_0 \), proving the claim.

So consider a sequence of double bubbles \( B_i \) of small areas \( A_i \geq A_i' > 0 \) converging to 0, which minimize perimeter among double bubbles of small areas at least \( A_i, A_i' \). We may assume that \( A_i'/A_i \to \lambda \). By the isoperimetric inequality, there is an \( \varepsilon_1 > 0 \) such that a connected component \( C_i \) of \( B_i \) maximizing the area of the first region has area at least \( \varepsilon_1 A_i \) of the first region and that similarly a component has (maximal) area at least \( \lambda \varepsilon_1 A_i \) of the second region. Let \( p_i, q_i \) be points of such components.

Move scalings of \( M \) by \( A_i^{-1/2} \) tangent to the \( x_1, x_2 \)-plane \( \Pi \) at the origin, with \( p_i \) at the origin. Henceforth we deal with such renormalizations. We may assume that \( C_i \), together with any other components within unit distance of \( C_i \), converge to a perimeter-minimizing bubble \( C \) in \( \Pi \), with first region area at least \( \varepsilon_1 \), necessarily a standard double bubble, or circle if the second area vanishes. For some \( \varepsilon_2 > 0 \), there are smooth families of very localized deformations of \( \mathbb{R}^N \) with \( \Delta A' = 0, |\Delta P/\Delta A| < 2\varepsilon_1^{-1/2} \), for \( |\Delta A| < \varepsilon_2 \). (A circle in \( \mathbb{R}^2 \) of area \( \varepsilon_1 \) has curvature \( dP/dA = (\pi/\varepsilon_1)^{1/2} < 2\varepsilon_1^{-1/2} \). For a standard double bubble, the curvature is still smaller.)
Case 1: \( \lambda > 0 \). If \( \lambda > 0 \), one similarly obtains \( \varepsilon_3 > 0 \) and deformations with \( \Delta A = 0 \), \( |\Delta P/\Delta A'| < 2(\lambda \varepsilon_1)^{-1/2} \), for \( |\Delta A'| < \varepsilon_3 \). Consequently, for \( i \) large, in the scaled bubbles \( B_i \), one may make arbitrary small adjustments \( \Delta A, \Delta A' \) in area at cost at most \( 3\varepsilon_1^{-1/2}|\Delta A| + 3(\lambda \varepsilon_1)^{-1/2}|\Delta A'| \) (first by a limit argument with some \( |\Delta A'| \ll |\Delta A| \) or some \( |\Delta A| \ll |\Delta A'| \), and hence for arbitrary small \( |\Delta A|, |\Delta A'| \) by linear combinations). There are two immediate consequences. First, the curvatures of the constant-curvature curves comprising the \( B_i \) are bounded by a single constant. Second, by the isoperimetric inequality, there are no very small components. We may assume that there are exactly \( m \) components. We may assume that each converges as above to a planar circle or standard double bubble. Minimization implies that there must be a single component converging (weakly) to the standard double bubble. Bounded curvature implies smooth convergence. Hence for large \( i \), \( B_i \) is standard, the desired contradiction.

Case 2: \( \lambda = 0 \). If \( \lambda = 0 \), \( C \) must be a circle. Any other component must converge to 0, because a limit with more than one circle would not minimize perimeter. For \( i \) large, unless \( C_i \) is a nearly round circle, the main component of the first region in \( C_i \) is a curvilinear polygon with \( 2s \geq 2 \) sides of alternating curvatures \( \kappa_1, \kappa_0 = \kappa_2 - \kappa_1 \) and 120-degree angles, as in Figure 1a, with \( \kappa_0 > 0 \). For the rest of \( C_i \) to fit together, \( C_i \) must be nearly a round component of the first region with small (nearly identical) bubbles of the second region, as in Figure 1b.

Figure 1
The scaled limit \( C \) must be a nearly round circle with small bubbles of the second region.
We need a bound on the number $m_i$ of little components of the second region. Since all are circles or digons of the same curvature, all have roughly the same perimeter and area, area at least $A_i'/2m_i$ and perimeter at least $\sqrt{A_i'/m_i}$ by the isoperimetric inequality. Let $Q_i$ denote the least perimeter to enclose area $A_i$. Since the interface between the first region and the exterior, together with the shorter side of each digon, encloses area at least $A_i$, those perimeters contribute at least $Q_i$. Therefore the total perimeter is at least

$$Q_i + \sum \sqrt{A_i'/m_i} = Q_i + \sqrt{m_iA_i'/4}.$$

One way to enclose areas $A_i, A_i'$ is with two near circles of perimeter less than $Q_i + 4A_i'$, as in Figure 2. Therefore $m_iA_i'/4 < 4A_i'$ and $m_i < 16$. Now we may assume that $C_i$ is a nearly round circle with a fixed number $m_1$ of bubbles of the second region and that the rest of $B_i$ is a fixed number $m_2$ of small nearly round bubbles of the second region. To remain minimizing in the limit, we must have $m_1 = 1$ and $m_2 = 0$, a standard double bubble, as desired.

Since the double bubble must have less perimeter than two separate bubbles, it follows easily from the isoperimetric inequality that the small region has fewer than 16 components.

2.2. Immiscible fluids. As suggested in [CF, Sect. 2.1], Theorem 2.1 generalizes to the immiscible fluids problem, in which interfaces between different regions (or exterior) carry different costs (see [M2, Chapt. 16]).
References


