Five Theorems in the theory of Riesz spaces

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When, in the beginning of the previous century, functional analysis offered a new perspective by focusing on vector spaces of functions rather than individual functions, order quickly developed as a good tool. Instead of integration of individual functions, the focus is on integration as a positive operator on a vector space of functions. Motivation for the development of positivity in functional analysis came from a variety of directions with important stimuli from the main theorems in functional analysis itself, from the theory of matrices with nonnegative entries, from measure theory and integral operators. One of the founders of functional analysis, F. Riesz, noticed how the individual decomposition of a real measure into a negative and a positive part resembled the order structure of the dual of the continuous functions on [0, 1]. A vector space $E$ that is partially ordered by a partial order denoted by $\leq$ is called a partially ordered vector space if for all $x, y, z$

\[
x \leq y \Rightarrow \alpha x \leq \alpha y \text{ if } \alpha \geq 0 \text{ and } x \leq y \Rightarrow x + z \leq y + z
\]

The subset of all elements $x$ of a partially ordered vector space $E$ for which $0 \leq x$ is denoted by $E^+$. A subset $A$ of $E$ is called order bounded if there exists an element $y$ in $E$ such that $x \leq y$ for all $x \in A$ and $y$ is then called an upper bound for $A$; if $A$ is order bounded then $z$ is called the least upper bound of $A$ if $z$ is an upper bound for $A$ and $z \leq y$ for every upper bound $y$ of $A$. Partially ordered vector spaces in which each pair of elements has a least upper bound are nowadays called Riesz spaces in honor of Riesz. The least upper bound of a set with two elements $x, y$ is denoted by $x \vee y$.

An element $x$ of $E$ is positive if $x \geq 0$. An easy consequence of these definitions is the fact that each element $x$ of a Riesz space $E$ can be decomposed as the difference of two positive elements, its positive part $x^+ = x \vee 0$ and its negative part $x^- = (-x) \vee 0$. The absolute value of $x$ is defined as

\[|x| = x \vee -x.\]

This paper is the residue of a talk on February 25, 2000 for the Louisiana-Mississippi sectional meeting of the MAA with the title Some Theorems in the Theory
of Riesz spaces. In that talk we aimed to motivate and invite the listener into the Theory of Positivity in Functional Analysis. To that end we chose a set of five theorems for which we hoped that the promotional effect was maximal. We realize that our choice of five theorems is subjective and apologize if the Reader’s Favorite Theorem is left out. Proofs are only sketched in one instance and deleted altogether otherwise. For a thorough introduction to Riesz spaces we refer to [11] or the more encyclopedic [12], while for the theory of positive operators we recommend to first read [14] and then the more advanced [1]. For connections between Riesz spaces and topological vector spaces see [3].

**Theorem 1: The positive part of a signed measure**

If \( E \) is a Riesz space and \( \phi : E \to \mathbb{R} \) is a linear functional that sends order bounded sets to order bounded sets then there exist two positive linear functionals \( \phi^+ \) and \( \phi^- \) such that \( \phi = \phi^+ - \phi^- \). Moreover

\[
\phi^+(y) = \sup\{\phi(x) : 0 \leq x \leq y\}
\]

for each \( y \in E^+ \). In fact, the set of all order bounded linear functionals on \( E \) is itself a Riesz space and the decomposition

\[
\phi = \phi^+ - \phi^-
\]

is exactly the decomposition of an element in its positive and negative part. In addition, the set of all order bounded linear functionals has the property that every nonempty order bounded subset has a least upper bound.

Theorem 1 above shows that every functional that sends order bounded sets to order bounded sets (a so-called order bounded functional) can be decomposed as the difference of two positive functionals. To get to the decomposition of a signed measure \( \mu \) as

\[
\mu = \mu^+ - \mu^-
\]

into a difference two positive measures form Theorem 1, one applies the Riesz Representation Theorem. Also note how the formula for computing the positive part of an order bounded functional resembles the formula for computing the positive part of a signed measure.

\[
\mu^+(B) = \sup\{\mu(A) : A \subset B\}.
\]

The property of Dedekind completeness which is expressed in the latter part of the Theorem is useful to lift many classical Theorems in functional analysis from the
setting of $\mathbb{R}$, the prime motivator for the study of the more abstract partially ordered vector spaces, to the setting of more general Riesz spaces. We first give the formal definition of Dedekind completeness and then an example of the latter phenomena.

A Riesz space $F$ is called Dedekind complete if every nonempty subset of $F^+$ has a least upper bound; it is called $\sigma$–Dedekind complete if every countable nonempty subset of $F^+$ has a least upper bound.

**Theorem 2: The Hahn-Banach Theorem**

Let $p$ be a sublinear function from the linear space $X$ to the Dedekind complete Riesz space $F$. Let $X_0$ be a vector subspace of $X$ and let $f_0$ be a linear function from $X_0$ to $F$ for which $f_0(x) \leq p(x)$ for all $x \in X_0$. Then there exists a linear function $f$ from $X$ to $F$ that extends $f_0$ and for which $f(x) \leq p(x)$ for all $x \in X$.

The beauty of the Theorem is this one: you can copy the Banach’s original proof of the Hahn-Banach Theorem, with $F$ instead of $\mathbb{R}$, word for word. In fact, Dedekind complete spaces are exactly the ordered vector spaces for which the Hahn-Banach extension works. Another proof of the Theorem works via a binary intersection property of balls and it is this property which has other connections with order. These connections enabled Nachbin to find that the injective Banach spaces are exactly those spaces of continuous functions on a compact Hausdorff space that are Dedekind complete. For other connections between the Hahn-Banach Theorem and positivity consult sections 6 and 21.2 in [5].

Sometimes a proof of a classical theorem can be mimicked when some additional machinery is added. A good example in case is the Cauchy-Schwarz inequality. Its generalization below was proved in [6], to which we refer for more details.

**Theorem 3: The Cauchy-Schwarz Inequality**

The classical Cauchy-Schwarz inequality states that, if $X$ is a vector space and $A : X \times X \to \mathbb{R}$ is a bilinear map with

$$
A(x, x) \geq 0 \quad \text{and} \quad A(x, y) = A(y, x)
$$

for all $x \in X$ then

$$
A(x, y)^2 \leq A(x, x)A(y, y).
$$

To have a Cauchy-Schwarz Inequality at all, we need to have a range space which besides an order structure also has an algebra structure. A Riesz space which is also an associative algebra is called a *lattice ordered algebra* if

$$
xy \geq 0
$$
whenever $x \geq 0$ and $y \geq 0$.

The usual proof of the Cauchy-Schwarz inequality starts off as follows. Take $x, y \in X$, $f_0 := A(x, x)$, $g_0 := A(x, y)$, $h_0 := A(y, y)$. From

$$A(x + \lambda y, x + \lambda y) \geq 0 \quad (\lambda \in \mathbb{R})$$

it follows that

$$f_0 + 2\lambda g_0 + \lambda^2 h_0 \geq 0 \quad (\lambda \in \mathbb{R}).$$

Then one looks at this as a quadratic in $\lambda$ and the inequality follows. We used function symbols $f, g,$ and $h$ to suggest that we are thinking of replacing $\mathbb{R}$ as a range space for $A$ with an algebra of functions, where the ordering and multiplication are pointwise. Such algebras are lattice-ordered. But what if the multiplication is not pointwise? G. Birkhoff started in 1956 a study of so-called almost f-algebras. An associative real lattice ordered algebra $E$ is called an almost f-algebra if

$$x \land y = 0 \Rightarrow xy = 0$$

for all $x, y \in E$. Of course Riesz algebras of functions under pointwise ordering and multiplication are the perfect examples of almost f-algebras. It turns out that the classical Cauchy-Schwarz inequality can be generalized to that setting. The main ideas are twofold. First of all, every Riesz space is isomorphic to a space of continuous functions. Secondly, the connection between the almost f-algebra multiplication and the pointwise multiplication can be easily understood as follows.

If $E$ is a Riesz space, a bilinear map $A$ of $E \times E$ into a vector space $F$ is called orthosymmetric if

$$f, g \in E, \quad f \land g = 0 \quad \Rightarrow \quad A(f, g) = 0.$$ 

The classical example of an orthosymmetric bilinear map is the multiplication on an almost f-algebra. Thus the following Lemma describes this connection between an almost f-algebra multiplication and the pointwise multiplication. The only new word that appears in it is the word Archimedean. A Riesz space $F$ is called Archimedean if for every $x, y \in F$ we have

$$0 \leq nx \leq y \text{ for all } n \in \mathbb{N} \Rightarrow x = 0.$$

Let $S$ be a compact Hausdorff space, $E$ a uniformly dense Riesz subspace of $C(S)$, $F$ an Archimedean Riesz space and $A$ an orthosymmetric positive bilinear
map $E \times E \to F$. Let $E^2$ be the linear hull of \{fg : f, g \in E\}. Then there exists an increasing linear map $B : E^2 \to F$ such that
\[ A(f, g) = B(fg) \quad (f, g \in C(S)). \]

The Lemma has some immediate consequences. For instance, an almost f-algebra is automatically commutative. Continuing our argument involving the Cauchy-Schwarz inequality we proceed as follows. Every Riesz space is isomorphic to a space of functions. Denoting the image of an element of the almost f-algebra $E$ by adding a hat to it, we have
\[ \hat{f}_0 + 2\lambda \hat{g}_0 + \lambda^2 \hat{h}_0 \geq 0 \quad (\lambda \in \mathbb{R}), \]
whence $\hat{g}_0 \hat{g}_0 \leq \hat{f}_0 \hat{h}_0$. Denoting the multiplication on $E$ by a fat dot, we wish to show that $g_0 \cdot g_0 \leq f_0 \cdot h_0$. Now apply the above Lemma to the bilinear map $(\hat{f}, \hat{g}) \mapsto f \cdot g$ of $\hat{F}_0 \times \hat{F}_0$ into $F$ and obtain an increasing linear $B : \hat{F}_0^2 \to F$ with
\[ f \cdot g = B(\hat{f} \cdot \hat{g}) \quad (f, g \in F_0). \]
Then $g_0 \cdot g_0 = B(\hat{g}_0 \hat{g}_0) \leq B(\hat{f}_0 \hat{h}_0) = f_0 \cdot h_0$.

Thus we have sketched a proof of the promised Theorem:

Let $X$ be a vector space and let $F$ be an almost f-algebra. If $A : X \times X \to F$ is a bilinear map with
\[ A(x, x) \geq 0 \quad \text{and} \quad A(x, y) = A(y, x) \]
for all $x \in X$ then
\[ A(x, y)^2 \leq A(x, x)A(y, y). \]

**Theorem 4: The Dodds-Fremlin Theorem**

The Dodds-Fremlin Theorem started a new development in positivity. The appearance of the main condition in the Theorem is not unlike the Radon Nikodym Theorem. Suppose $S, T$ are positive operators $E \to F$. Suppose that
\[ 0 \leq S \leq T \]
and that $T$ has a "nice property". Does $S$ inherit this nice property under suitable conditions on the spaces $E, F$? The answer is yes for many nice properties and mild acceptable conditions on the spaces $E, F$. The property under investigation by Dodds and Fremlin is compactness and thus we want our spaces $E, F$ to be Banach spaces. A Riesz space $E$ on which there is a norm $\|\cdot\|$ that renders the Riesz space a Banach space is called a Banach lattice if

$$\|x\| \leq \|y\| \Rightarrow \|x\| \leq \|y\|$$

for all $x, y \in E$. The Banach lattice $E$ is said to have order continuous norm if each order interval $\{z : x \leq z \leq y\}$ is weakly compact. Of course every $L^2$ has order continuous norm. The Banach dual of $F$ is denoted by $F^*$. Here is the Theorem.

(Compact Domination Theorem, Dodds-Fremlin) If $E$ and $F^*$ have order continuous norm and $S, T$ are positive operators $E \to F$ for which

$$0 \leq S \leq T$$

then compactness of $T$ implies compactness of $S$.

By now, weak compactness of $T$, Dunford Pettis operators $T$ and others have been investigated in this so called domination theorem. When Dodds and Fremlin proved their Theorem about compact positive operators in [10], the result was not even known for $L^2$ and the question had come up in Mathematical Physics in a paper by Avron, Herst and Simon in [4]. The result has generated tremendous interest in positive operators. Both, the Dodds-Fremlin Theorem as well as the techniques that were developed as a consequence have made the field blossom since. The similarity with the Radon-Nikodym Theorem can be brought a little more to the fore (but it remains only a similarity to "multiplying" a measure with a function) by introducing an analogue of multiplication operators $\pi$ on $E$, $\sigma$ on $F$ and approximating the operator $S$ by operators of the type $\sigma \circ T \circ \pi$. For the latter approach we refer to [8]. It is also interesting to note that without any conditions on the spaces $E$ and $F$, the operator $S^3$ is compact if $T$ is compact and $0 \leq S \leq T$ (see [2]).

**Theorem 5: The de Pagter’s Theorem**

A Riesz space, like any vector space, can be complexified. This only makes sense if one can construct a modulus at the same time. In fact, one can use functional calculus (see [7]) to make sense of

$$\sqrt{x^2 + y^2}$$

for $x, y \in E$, if for instance $E$ is a Banach lattice. This then is defined to be the modulus of $x + iy \in E + iE$ and $E + iE$ is called a complex Banach lattice, in case
that $E$ is a Banach lattice. We need these complex Banach lattices to be able to talk about the spectrum of operators. We also need the notion of ideals. A complex vector subspace $I$ of the complex Banach lattice $E$ is called an ideal of $E$ if

$$|x| \leq |y| \text{ and } y \in I \Rightarrow x \in I.$$ 

An operator $T : E \to E$, where $E$ is a Banach lattice, is called ideal irreducible if the only invariant norm closed ideals are $\{0\}$ and $E$. De Pagter in [13] beautifully adapted the main argument in Hilden’s proof of the Lomonosov Invariant Subspace Theorem to prove the following result.

(de Pagter) Let $E$ be a complex $\sigma-$Dedekind complete Banach lattice and let $T$ be a positive compact ideal irreducible operator in $E$. If $T \neq 0$ then the spectral radius of $T$ is strictly positive.

Surely, the reader will feel the need to combine de Pagter’s result with the following Krein-Rutman Theorem.

(Krein-Rutman) Let $T$ be a positive compact operator on a complex Banach lattice $E$. If the spectral radius $r$ of $T$ is strictly positive then $r$ is an eigenvalue of $T$ and there exists a positive $x \in E$ such that $T(x) = rx$.

A particular consequence of de Pagter’s Theorem, when combined with the Krein-Rutman Theorem is the following Perron-Frobenius Theorem about positive matrices.

(Perron-Frobenius) If $T$ is a positive irreducible $n \times n$ matrix, i.e. all its entries $t_{ij}$ are nonnegative and for every decomposition into nonempty sets $X_1 \cup X_2 = \{1, 2, \ldots, n\}$ we have

$$\sum_{j \in X_1} \left( \sum_{i \in X_2} t_{ij} \right) > 0,$$

then the spectral radius $r$ of $T$ is strictly positive and $r$ is an eigenvalue for $T$ with an eigenvector having all of its entries strictly positive.

For a proof of the existence of a strictly positive eigenvalue in the Perron-Frobenius Theorem for $3 \times 3$ matrices by using Brouwer’s fixed point Theorem, we advise to work exercise 4F in [9]. Indeed, classical theorems outside the theory of Hilbert spaces that guarantee the existence of eigenvalues for a broad class of operators are rare and very useful. Both, the Krein-Rutman and de Pagter’s Theorem are important in applied mathematics.

We hope you have enjoyed this glimpse into the theory of Riesz spaces. Thank you for your interest.

REFERENCES

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