The Behavior of Fractions with Prime Denominators

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Abstract
Much of mathematics is the observation and manipulation of different patterns that can occur when dealing with different sets of numbers, shapes, etc. One of the most peculiar patterns that can be observed is how the decimal expansions of fractions behave whenever there is a prime denominator. One of the first observations that one can make is how full period primes behave in certain bases. We can then observe that the patterns that occur with full period primes occur with prime denominators in general.
The Behavior of Fractions with Prime Denominators

One of the strangest patterns that occurs is in the nonterminating decimal expansions of fractions with certain denominators. For example, consider the different fractions with 7 as the denominator:

\[
\frac{1}{7} = .142857 \ldots, \quad \frac{2}{7} = .285714 \ldots, \quad \frac{3}{7} = .428571 \ldots, \quad \frac{4}{7} = .571428 \ldots, \quad \frac{5}{7} = .714285 \ldots, \quad \frac{6}{7} = .857142 \ldots.
\]

They have the same block of repeating digits with a different digit starting the decimal expansion. This pattern is a result of 7 having the cycling digits property. More precisely, the series of repeating digits in a fraction, \(\frac{a}{b}\), that has a nonterminating decimal expansion is called the repetend of \(\frac{a}{b}\) (e.g., 142857 is the repetend of \(\frac{1}{7}\)). As mentioned before, 7 has the cycling digits property. An integer \(b\) has the cycling digits property in base \(\beta\) if every fraction with \(b\) as the denominator has a repetend that is a cyclic permutation of the repetend of \(\frac{1}{b}\). Note that in following, this property will be abbreviated as CDP. The number of digits in a repetend is called the period of the repetend. For example, the period of \(\frac{1}{7}\) is 6.

There is a way to visually represent the various fractions of prime integers and the digits that start their decimal expansions. This visual representation is called a circle diagram. For a circle diagram to be generated, however, the following conditions must be fulfilled: the integer that is in the denominator of the various fractions must be prime, the integer cannot be a factor of the base or the base minus one, the fractions have a numerator that is strictly less than the integer and strictly greater than zero, and no fractions repeat. For now, we will only consider the circle diagrams of integers with the cycling digits property. The circle diagram for the fractions of 7 in base 10 is shown in Figure 1:

![Figure 1. Circle Diagram for 7 in Base 10](image)

The circle diagram is generated by writing out the decimal expansions of all of the fractions with 7 as the denominator. Then the fractions are then ordered in a way such that a truncated left shift is shown in their decimal expansions. So, for example, start out with the decimal expansion for \(\frac{1}{7} = .142857 \ldots\). The fraction that goes into the second spot of the circle diagram going clockwise is the fraction that has the decimal expansion 0.428571 \ldots which is \(\frac{3}{7}\). This process is repeated and once the original decimal expansion is obtained, then the circle diagram is complete.

We can find the remainders of the long division process when \(a\) is divided by \(b\) in a recursive way. Consider the circle diagram for 7 in base 10 (Figure 1). Label the numerators along the circle diagram starting with 1 going clockwise as \(r_1 = 1, r_2 = 3, r_3 = 2, r_4 = 6, r_5 = 4,\) and \(r_6 = 5\). Notice that \(r_2 = 3 \equiv 10 \equiv 1 \cdot 10 \equiv r_1 \cdot 10 \pmod{7}\). Similarly, \(r_3 = 2 \equiv 30 \equiv 3 \cdot 10 \equiv r_2 \cdot 10 \pmod{7}\). In general, \(r_i \equiv r_{(i-1)} \cdot 10 \pmod{7}\) where \(1 \leq i \leq 6\) and \(i\) denotes the place in which the numerator is located on the circle diagram with respect to 1. More generally: \(r_i \equiv 10^{(i-1)} \pmod{7}\) where \(1 \leq i \leq 6\) and \(i\) denotes the place in which the numerator is located on the circle diagram. Further generalizing to an arbitrary base: \(r_j \equiv \beta^{(j-1)} \pmod{b}\) where \(1 \leq j \leq k\) where \(k\) is the period of the repetend of \(\frac{1}{b}\) in base \(\beta\) and \(j\) denote the place in which the numerator is located on the circle diagram for \(b\) in base \(\beta\). This will be referred to as the recursive way to find remainders.

At this point, it should be noted that 2 is never considered to be an integer with the cycling digits property in any base. It will be shown later that in order for an integer to have the cycling digits property in any base, it cannot be a factor of the base or the base minus one. 2 will be a factor of the base if the base is even, and 2 will be a factor of the base minus one if the base is odd. Therefore, it is always necessary to exclude 2 when proving the various characteristics about digits with the cycling digits property.
Consider Figure 1. The digits inside the circle that are directly across from each other have a sum of 9. Additionally, the fractions that are directly across from each other have a sum of 1. Consider the circle diagram for 17 in base 10.

Figure 2. Circle Diagram for 17 in Base 10

The same pattern occurs with 17 because it has the cycling digits property in base 10. Now consider the circle diagram that can be generated for 13 in base 2.

Figure 3. Circle Diagram for 13 in Base 2

A similar pattern occurs; however, instead of having a sum of 9, the digits inside the circle directly across from each other have a sum of 1. Notice that 9 = 10 − 1 and 1 = 2 − 1. In either case, the sum of digits that are diametrically located across from each other have a sum of the base minus one. Let this pattern be denoted as Property One. Now notice that even though circle diagrams behave differently in different bases when looking at the sum of digits inside the circle, the fractions outside the circle that are located diametrically across from each other have a sum of 1 regardless of base. Denote the pattern that opposing fractions have a sum of 1 as Property Two. To prove these properties, other theorems dealing with numbers with the cycling digits property need to be proven.

As aforementioned, 7 is an integer with the cycling digits property in base 10 and every fraction with 7 as the denominator has a repetend that has the repeating block of digits 142857. The next two integers that have the cycling digits property in base 10 are 17 and 19.

\[
17 = 0.0588235294117647\ldots
\]

and

\[
19 = 0.052631578947368421\ldots
\]

If you look at the repetends of \( \frac{1}{7} \), \( \frac{1}{17} \), and \( \frac{1}{19} \) in base 10, the period of the each repetend is equal to the denominator minus one. This is also true for \( \frac{1}{13} \) in base 2. This leads to the conjecture that states the following: if \( b \) has the cycling digits property \( \iff \frac{1}{b} \) has a repetend with a period of \( b - 1 \). To prove this result, the following lemma is needed:

**Lemma 1.** If \( \frac{1}{n} \) has a nonterminating decimal expansion in base \( \beta \), then the repetend of \( \frac{1}{n} \) will have a maximum period of \( n - 1 \).

**Proof.**
Let \( n \in \mathbb{Z} \) such that \( \frac{1}{n} \) has nonterminating decimal expansion in base \( \beta \). Consider the decimal expansion of \( \frac{1}{n} \). Each step of the algorithm for long division in base \( \beta \) must produce a nonzero remainder because \( \frac{1}{n} \) has a nonterminating decimal expansion. There are \( n - 1 \) possible nonzero remainders. Once a remainder is repeated, the division process will repeat itself. Thus, the repetend of \( \frac{1}{n} \) will have maximum period of \( n - 1 \).

**QED**

Now we can prove the conjecture above.

**Theorem 1.** \( b \) has the cycling digits property \( \iff \frac{1}{b} \) has a repetend with a period of \( b - 1 \).

**Proof.**
\( \Rightarrow b \in \mathbb{Z} \) with the CDP is assumed.
By Lemma 1, we know that the repetend of \( \frac{1}{b} \) will have a maximum period of \( b - 1 \). Since \( b \) has the CDP,
∀a ∈ Z, with 1 ≤ a ≤ b − 1, the fraction \( \frac{a}{b} \) has a repetend that is a cyclic permutation of the repetend of \( \frac{1}{b} \). There are \( b−1 \) such fractions. For each permutation to be distinct, the period of the repetend of \( \frac{1}{b} \) must be at least \( b−1 \). Thus, the period of the repetend of \( \frac{1}{b} \) is \( b−1 \).

\[ \iff \frac{1}{b} \text{ has a repetend with a period of } b−1 \text{ is assumed.} \]

Then through the long division process of \( \frac{a}{b} \), each of the possible \( b−1 \) nonzero remainders occur once before 1 is repeated is the process of long division. Consider the long division process for \( \frac{a}{b} \) for \( 1 < a ≤ b−1 \). Since \( a \) is one of the possible \( b−1 \) nonzero remainders, when \( a \) is divided by \( b \), \( \frac{a}{b} \) will have a similar long division process to \( \frac{1}{b} \), so long division process for \( \frac{a}{b} \) is done in the same manner as \( \frac{1}{b} \), but the process begins at \( a \). Therefore, \( \frac{a}{b} \) will have a repetend that is a cyclic permutation of \( \frac{1}{b} \). Thus, \( a \) has the CDP.

\[ \text{QED} \]

Now, consider the numbers that have been shown to have the cycling digits property so far (7, 17, and 19 in base 10, and 13 in base 2). Each number that has been shown is a prime number. In general we can show that if an integer \( b \) has the cycling digits property, then \( b \) is prime. The proof for Theorem 2 goes as follows:

**Theorem 2.** If \( b \) is an integer with the CDP, then \( b \) is prime.

**Proof.**

Let \( b ∈ Z \) with \( b \) having the CDP. Suppose towards a contradiction that \( b \) is not prime. Then \( ∃q, k ∈ Z \) with \( 1 < q, k < b \) such that \( b = qk \). Consider \( \frac{a}{b} \). Notice that \( \frac{a}{b} = \frac{a}{qk} = \frac{1}{k} \). By Lemma 1, we know \( \frac{1}{k} \) can have a repetend with a maximum period of \( k−1 \). By Theorem 1, \( \frac{a}{b} \) will have a repetend with a period of \( b−1 \). So, \( b−1 ≤ k−1 \implies b ≤ k \), but this is a contradiction since \( k < b \). Thus, \( b \) is prime.

\[ \text{QED} \]

From the results of Theorem 2, we know that in order for an integer \( b \) to have the cycling digits property in base \( β \), it must be prime; however, not all prime numbers have the cycling digits property in every base (i.e 5 in base 10). To distinguish between prime numbers with and without the cycling digits property in base \( β \), we will call primes that have the cycling digits property in base \( β \) full period primes.

Recall the technique for finding the fraction form of a nonterminating repeating decimal. The technique goes as follows: Call the nonterminating repeating decimal \( N \). If there are \( n \) digits in the repetend of \( N \), then multiply the decimal expansion by \( 10^n \). Then, subtract \( N \). This will give you a whole number representation of the repetend of \( N \). Divide this whole number representation of the repetend by \( 10^n−1 \). This will give the fraction form of \( N \). This technique also works in arbitrary bases. A very similar process is followed, however, instead of multiplying by \( 10^n \), multiply by \( β^n \) where \( β \) is the base. Using this technique, we can prove further results about full period primes.

**Lemma 2.** Let \( x ∈ Z \) with \( x ≠ 0 \). Let \( n ∈ N \). Consider \( \frac{1}{x} \) in base \( β \). \( (β^n−1)\frac{1}{x} ∈ Z \iff n \) is a multiple of the period of \( \frac{1}{x} \).

**Proof.**

\[ \implies (β^n−1)\frac{1}{x} ∈ Z \text{ is assumed.} \]

Let \( \frac{1}{x} = .d_1d_2...d_kd_1... \) where each \( d_i \) is a digit such that \( 0 ≤ d_i ≤ β−1 \) and \( k \) is the period of \( \frac{1}{x} \). In base \( β \),

\[ β^n(\frac{1}{x}) = d_1d_2...d_n.d_{n+1}... \]

Thus, \( d_1 = d_{n+1}, d_2 = d_{n+2} \). So, \( n = hk \) for some \( h ∈ N \).

\[ \iff \frac{1}{x} \text{ has a repetend whose period is } k \text{ and } n = hk \text{ for some } h ∈ N \text{ is assumed.} \]

Consider \( \frac{1}{x} \) in base \( β \). Then,

\[ \frac{β^n}{x} = \frac{(β^n−1)\frac{1}{x}}{x} = (β^{hk})\frac{1}{x}−\frac{1}{x} \]

\[ β^n(\frac{1}{x}) = d_1d_2...d_{k}d_1...d_{n}d_{n+1}...d_{k}d_1...d_{k}d_1...d_kd_1...d_k ∈ Z \]

\[ h \text{ groups} \]

\[ h \text{ groups} \]

\[ \text{QED} \]

**Lemma 3.** If \( b \) is a full period prime in base \( β \) and \( γ = β−1 \), then \( b \nmid γ \).

**Proof.**

Let \( b \) be a full period prime in base \( β \). Let \( γ = β−1 \). Suppose towards a contradiction that \( b \mid γ \). Then \( ∃w ∈ Z \) such that \( bw = γ = β−1 \). So, \( w = (β^1−1)\frac{1}{b} \). By Lemma 2, 1 is a multiple of the period of \( \frac{1}{b} \). This is a contradiction because by Theorem 1, \( \frac{1}{b} \) has a period of \( b−1 \). Thus, \( b \nmid γ \).

\[ \text{QED} \]
Using the results of Lemma 2 and Lemma 3, we can prove Theorem 3:

**Theorem 3.** $b$ is a full period prime in base $\beta$ $\iff b|111\ldots1$ in base $\beta$ where there are $b-1$ ones, but $b \nmid 111\ldots1$ with fewer than $b-1$ ones.

**Proof.**

$\Rightarrow b$ is a full period prime in base $\beta$ is assumed.

Let $\gamma = \beta - 1$. By Theorem 1, we know that $\frac{1}{\beta}$ has a repetend with a period of $b - 1$. By Lemma 2, $(\beta^{(b-1)} - 1)\frac{1}{\beta} \in \mathbb{Z}$. Let $r = (\beta^{(b-1)} - 1)\frac{1}{\beta}$. Then, $rb = \beta^{(b-1)} - 1$. Thus, $b|(\beta^{(b-1)} - 1)$. Consider

$$\beta^{(b-1)} - 1 = \underbrace{\gamma\gamma\ldots\gamma}_{b-1} = \gamma(111\ldots1).$$

So, $b|\gamma111\ldots1_{b-1}$. By Lemma 3, $b \nmid \gamma$ and $b$ is prime so $b|111\ldots1_t$. Suppose towards a contradiction that $b|111\ldots1_t$ for some $t \in \mathbb{N}$ such that $t < b - 1$. Then, $b|\gamma(111\ldots1_t)$. But, $\gamma(111\ldots1_t) = \beta^t - 1$. So, $b|\beta^t - 1$. Then, $\exists u \in \mathbb{Z}$ such that $bu = \beta^t - 1$. Then, $bu = \beta^t - 1 \implies u = (\beta^t - 1)\frac{t}{b}$. Therefore, $t$ is a multiple of the period of the repetend of $\frac{1}{b}$ by Lemma 2. So, $t = (b - 1)n$ for some $n \in \mathbb{Z}$. This is a contradiction because $t < b - 1$. Thus, $b|111\ldots1_t$ where $t < b - 1$.

$$\Leftarrow b|111\ldots1_t \text{ but } b\nmid 111\ldots1_t \text{ with } 1 \leq t < b - 1 \text{ in base } \beta \text{ is assumed.}$$

Then $b|\gamma111\ldots1_{b-1}$. So, $\gamma111\ldots1_{b-1} = \gamma\gamma\ldots\gamma = \beta^{b-1} - 1$. Thus, $b|\beta^{(b-1)} - 1$ but $b \nmid \beta^{t} - 1$. Then $\exists d \in \mathbb{Z}$ such that $bd = \beta^{b-1} - 1$. So, $d = (\beta^{(b-1)} - 1)\frac{1}{b}$ but $\frac{2}{b}w \in \mathbb{Z}$ such that $w = (\beta^t - 1)\frac{t}{b}$. By Lemma 2, the period of the repetend of $\frac{1}{b}$ is a multiple of $b - 1$. Since $(\beta^t - 1)\frac{t}{b} \notin \mathbb{Z}$, $t$ is not a multiple of the period. So $\frac{1}{b}$ has a repetend with a period of $b - 1$. By Theorem 1, $b$ is a full period prime.

**QED**

In his article, Dan Kalman defines a number of the form $111\ldots1$ as a repunit (repeated unit) [1]; however, for our purposes, this definition is insufficient since the term repunit implies that a number of this form consists only of the digit 1. So we will use a similar idea to make the following definition: a repdigit in base $\beta$ is a number of the form $b\ldots b$ where $b < \beta$.

Using the results of the first three theorems, it is possible to prove Property One of the circle diagram. Recall that the first property of the circle diagram stated that each pair of digits that were directly across from each other had a sum of the base minus one. Theorem 4 proves this property to be true for all full period primes in any arbitrary base. Before proving Theorem 4, notice that the repetend of $\frac{1}{b}$ has a period that is even because $b$ is a prime number and 2 is excluded. Knowing this, Property One can be rewritten to state the following: Consider the circle diagram for $b$ in base $\beta$. Let $k = \frac{b - 1}{2}$. Each pair of digits in the repetend of $\frac{1}{b}$ that are a distance of $k$ digits apart have a sum of the base minus one. It is also important to note that similar to how $.999\ldots = 1$ in base 10, $\gamma\gamma\ldots\gamma = 1$ in base $\beta$ if $\gamma = \beta - 1$. The proof of Property One goes as follows:

**Theorem 4.** Let $b$ be a full period prime in base $\beta$. If the repetend of $\frac{1}{b}$ is the sequence of digits $(d_1, d_2, \ldots, d_{2k})$, with $1 \leq d_t < \beta$, then $d_1 + d_{k+1} = d_2 + d_{k+2} = \ldots = d_k + d_{2k} = \gamma$.

**Proof.**

Let $b$ be a full period prime in base $\beta$. By Theorem 1, $\frac{1}{b}$ has a period of $b - 1$. Since $b$ is odd, $\frac{1}{b}$ has a period that is even. So, $\exists k \in \mathbb{Z}$ such that $b - 1 = 2k$. By Theorem 3, $b|111\ldots1_k$ but $b \nmid 111\ldots1_{2k}$. Consider $(\beta^k + 1)(111\ldots1_k)_{2k}$ in base $\beta$.

$$(\beta^k + 1)(111\ldots1_k)_{2k} = \frac{1}{b}(\beta^{(b-1)} - 1)\frac{1}{\beta} = \beta^{(b-1)} - 1 = \underbrace{\gamma\gamma\ldots\gamma}_{b-1} = \gamma(111\ldots1).$$

So $b|(\beta^k + 1)(111\ldots1_k)$. Since $b$ is prime and $b \nmid (111\ldots1_k)$, $b|\beta^k + 1$. So, $\exists g \in \mathbb{Z}$ such that $bg = \beta^k + 1$. Now, $g = (\beta^k + 1)\frac{1}{b}$.

$$\frac{1}{b} = d_1d_2\ldots d_{2k} \ldots$$

$$\beta^k\frac{1}{b} = d_1d_2\ldots d_kd_{k+1}d_{k+2}\ldots d_{2k} \ldots$$

$$\beta^k\frac{1}{b} + \frac{1}{b} = d_1d_2\ldots d_kd_{k+1}d_{k+2}\ldots d_{2k} + d_1d_2\ldots d_{2k} \ldots$$
Because \((\beta^{k+\frac{1}{2}} + \frac{1}{2}) \in \mathbb{Z}\) and both terms of the sum are nonterminating decimals in base \(\beta\), 
\[d_{k+1} + d_1 = d_{k+2} + d_2 = \ldots = d_{2k} + d_k = \gamma.\]  
\[QED\]

To prove Property Two, some ideas from group theory are needed. The following definitions will be used: 
- definition of a group, subgroup, a cyclic subgroup, an abelian group, order of a group, order of an element, coset, and index. 
- There is also a well-known theorem from group theory that states that if \(p\) is prime, then the set of nonzero elements in \(\mathbb{Z}_p\) forms an abelian group under multiplication. 
- We denote this set \(\mathbb{Z}_p^*\). The following theorem will be used to prove Property Two and goes as follows:

**Theorem A.** Let \(G\) be a group with \(a \in G\). If \(a\) has a finite order \(m\), then \(|\langle a \rangle| = m\) and \(\langle a \rangle = \{a^0 = e, a, a^2, \ldots, a^m\}\)

Consider the multiplicative group of \(\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}\). Now consider \(\langle 10 \rangle\). \(\langle 10 \rangle = \{1, 3, 2, 6, 4, 5\}\). So \(\mathbb{Z}_7^* = \langle 10 \rangle\) and, 10 is a generator of \(\mathbb{Z}_7^*\). Similarly, 10 is a generator of \(\mathbb{Z}_{17}^*\). Now consider \(\mathbb{Z}_{13}^*\). 2 is a generator of \(\mathbb{Z}_{13}^*\). This pattern is proven in Theorem 5.

**Theorem 5.** \(b\) is a full period prime in base \(\beta \iff \beta\) is a generator of \(\mathbb{Z}_b^*\).

**Proof.**

\[\Rightarrow b\) is a full period prime in base \(\beta\) is assumed.

Consider \(\mathbb{Z}_b^* = \{1, 2, \ldots, b - 1\}\). Notice \(|\mathbb{Z}_b^*| = b - 1\). \(|\beta| \not= b - 1\) because \(|\mathbb{Z}_b^*| = b - 1\) and \(\beta \in \mathbb{Z}_b^*\). Consider the long division process for \(\frac{1}{b}\) in base \(\beta\). Recall the recursive way to find remainders. So, \(r_j \equiv \beta^{j-1} \pmod{b}\). 

\[r_{b-1} \equiv \beta^{b-2} \equiv \beta^{b-1} \pmod{b}\]. Thus, \(|\beta| < b - 1\) since there are \(b - 1\) remainders. So, \(|\beta| = b - 1\). By Theorem A, 
\[|\langle \beta \rangle| = b - 1\] and \(\langle \beta \rangle = \{\beta, \ldots, \beta^{b-1}\} = \mathbb{Z}_b^*\).

\[\Leftarrow \beta\) be a generator of \(\mathbb{Z}_b^*\) is assumed.

So, \(\langle \beta \rangle = \mathbb{Z}_b^* = \{1, 2, \ldots, b - 1\}\). Then \(\forall a \in \mathbb{Z}_b^*, \exists i \in \mathbb{N}\) such that \(\beta^i \equiv a \pmod{b}\). Consider the long division process for \(\frac{1}{b}\) in base \(\beta\). Recall the recursive way to find remainders. So, \(r_j \equiv \beta^{j-1} \pmod{b}\). 
\[|\langle \beta \rangle| = b - 1\], so \(|\beta| = b - 1\). So \(r_b \equiv \beta^{b-1} \equiv 1 \pmod{b}\) and \(r_k \neq 1\) for \(k < b - 1\). Note that \(\frac{b}{j} \in \mathbb{N}\) such that \(\beta^j \equiv 0 \pmod{b}\) because \(\langle \beta \rangle = \mathbb{Z}_b^*\). Thus, \(\frac{1}{b}\) is a nonterminating decimal in base \(\beta\) with a repetend that has a period of \(b - 1\). By Theorem 1, \(b\) is a full period prime.

**QED**

Using the theorems above, Property Two of the circle diagram can now be proven. Recall that Property Two states that pairs of fractions that are \(k\) places apart on the unit circle will have a sum of 1. To prove this, it is only necessary to consider the numerators of \(\frac{\beta}{k}\) with \(1 \leq a \leq b - 1\). Property Two is now proven:

**Theorem 6.** Let \(b\) be a full period prime in base \(\beta\). Let \(r_i \in \langle \beta \rangle\) with \(1 \leq i \leq b - 1\). Then \(r_1 + r_{k+1} \equiv r_2 + r_{k+2} \equiv \ldots \equiv r_k + r_{2k} \equiv 0 \pmod{b}\).

**Proof.**

Let \(b\) be a full period prime in base \(\beta\). By Theorem 5, \(\mathbb{Z}_b^* = \langle \beta \rangle\). We know that \(\frac{1}{b}\) has a repetend whose period is even. So, \(3k \in \mathbb{Z}\) such that \(b - 1 = 2k\). Recall the recursive way to find remainders. So, \(r_j \equiv \beta^{j-1} \pmod{b}\). Consider \(r_m + r_{m+k}\) in \(\mathbb{Z}_b^*\).

\[r_m + r_{m+k} \equiv \beta^{m-1} + \beta^{m+k-1} \equiv \beta^{m-1}(1 + \beta^k) \pmod{b}\].

By Theorem 3, \(b | 111 \ldots 1\) but \(b \nmid 111 \ldots 1\). Now, \(111 \ldots 1 = (\beta^k + 1)(111 \ldots 1)\). Since \(b\) is prime and \(b \nmid 111 \ldots 1\), 
\[b | (\beta^k + 1), \text{ so } \beta^k + 1 \equiv 0 \pmod{b}\]. Now, \(\beta^{m-1}(1 + \beta^k) \equiv \beta^{m-1}(0) \equiv 0 \pmod{b}\). Thus, \(r_1 + r_{k+1} \equiv r_2 + r_{k+2} \equiv \ldots \equiv r_k + r_{2k} \equiv 0 \pmod{b}\).

**QED**

Up to this point, only full period primes have been considered, but similar patterns can occur when \(b\) is not a full period prime. Consider \(\frac{1}{13}\) in base 10 (\(\frac{1}{13} = 0.076923 \ldots\)). So, the period of the repetend of \(\frac{1}{13}\) is 6 and 13 is not a full period prime by Theorem 1; however, there are 5 other fractions of 13 that have repetends that are cyclic permutations of the repetend of \(\frac{1}{13}\). These 5 fractions along with \(\frac{1}{13}\) can create a circle diagram. Now consider \(\mathbb{Z}_{13}\). As expected, \(|\langle 10 \rangle| = 6\). Consider \(\langle 10 \rangle = \{1, 10, 9, 12, 3, 4\}\). \(\langle 10 \rangle\) gives the set that contains all the numerators used in the circle diagram for \(\frac{1}{13}\) in base 10. Now consider the coset \(2\langle 10 \rangle = 2, 7, 5, 11, 3, 8\). Using the elements of this coset as numerators, a second circle diagram can be generated. Figure 5 shows this circle diagram. Both of these circle diagrams exhibit the same properties as the circle diagrams for full period primes.
which leads one to believe that this is true for all circle diagrams that can be generated regardless of whether or not \( b \) is a full period prime.

Before proving these properties to be true, it is necessary to prove some other properties about fractions with prime denominators. Notice that \([\mathbb{Z}^*_b : \langle \beta \rangle] = 2\) and the period of the repetend of \( \frac{1}{b} \) is 6. Also, \(|\mathbb{Z}^*_b| = 12 = (2)(6) = [\mathbb{Z}^*_b : \langle \beta \rangle] \cdot |\langle \beta \rangle|\). This means that even though the period of the repetend of \( \frac{1}{b} \) is not 12 = 13 − 1 in base 10, the repetend of \( \frac{1}{b} \) in base 10 divides 12. This pattern is proven for all prime numbers in Theorem 7, but before Theorem 7 can be proven, a couple of theorems from abstract algebra are needed. The two theorems that will be used are:

LaGrange’s Theorem. Let \( K \) be a subgroup of a finite group \( G \). If \( |K| = m \) and \( |G| = n \), then \( m|n \) and \( n = m \cdot \lceil G : K \rceil \).

Theorem B. Let \( G \) be a group. Let \( H \) be a subset of \( G \) with \( H \neq \emptyset \). If \( H \) satisfies the following two conditions, then \( H \) is a subgroup of \( G \):

1. If \( a, b \in H \), then \( ab \in H \).
2. \( \forall a \in H, \exists a^{-1} \in H \)

We are now ready to prove this pattern to be true:

Theorem 7. Let \( b \) be a prime number. Then the period of the repetend of \( \frac{1}{b} \) in base \( \beta \) divides \( b - 1 \) and \(|\langle \beta \rangle| \cdot [\mathbb{Z}^*_b : \langle \beta \rangle] = |\mathbb{Z}^*_b|\).

Proof. Let \( b \) be a prime number. Consider \( \mathbb{Z}^*_b \). Let \( |\beta| = m \) for \( m \in \mathbb{Z} \). Consider \( \langle \beta \rangle = \{\beta^0, \beta^1, \ldots, \beta^{m-1}\} \). Consider \( \beta^n, \beta^p, \beta^{-n} \in \langle \beta \rangle \) with \( n, p \in \mathbb{Z} \). \( \beta^n \beta^p = \beta^{n+p} \in \langle \beta \rangle \). \( \beta^n \beta^{-n} = \beta^{n-n} = \beta^0 = 1 \). Thus, by Theorem B \( \langle \beta \rangle \) is a subgroup of \( \mathbb{Z}^*_b \). By Theorem A, \(|\langle \beta \rangle| = m \). We know that \( |\mathbb{Z}^*_b| = b - 1 \). Then, by LaGrange’s Theorem, \( m \cdot [\mathbb{Z}^*_b : \langle \beta \rangle] = |\mathbb{Z}^*_b| = b - 1 \) and \( m|b - 1 \).

\( QED \)

Most theorems that have been shown and that will be shown were proven in base 10 and then generalized to arbitrary bases. The remaining theorems are also generalizations of the results from base 10. Consider the circle diagrams shown in figures 1-5. Notice that the digits of the repetend around the circle diagram have a sum that is divisible by the base minus one. This is true for all primes in base 10 with the exception of 2, 3, and 5. Thus, when proving Theorem 10 it was necessary to restrict the proof to primes greater than 5. Theorem 8 and Theorem 9 show why this condition was necessary and give necessary conditions to prove this for any arbitrary base. Notice that 2 and 5 are prime factors of 10. Let \( A = \{\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\} \) for fractions in base 10. \( A \) is a set that only contains fractions with terminating decimal expansions. A similar pattern occurs when looking
at the fractions of the form $\frac{a}{b}$, i.e. fractions with 7 as the denominator terminate in base 14. This pattern led to Theorem 8:

**Theorem 8.** Let $b \in \mathbb{Z}$. If $\beta$ is a multiple of $b$, then $\frac{1}{b}$ has a terminating decimal expansion in base $\beta$.

**Proof.**
Let $b \in \mathbb{Z}$. Consider base $\beta$. Let $\beta$ be a multiple of $b$. Then $\exists h \in \mathbb{Z}$ such that $\beta = bh$. Suppose towards a contradiction that $\frac{1}{b}$ has a nonterminating decimal expansion. Let $\frac{1}{b}$ have a repetend whose period is $k$. By Lemma 2, $(\beta^k - 1)\frac{1}{b} \in \mathbb{Z}$. Consider

$$(\beta^k - 1)\frac{1}{b} = \beta^k\frac{1}{b} - \frac{1}{b} = (bh)^k\frac{1}{b} - \frac{1}{b} = b^{k-1}h^k - \frac{1}{b}.$$  

This is a contradiction because $b^{k-1}h^k - \frac{1}{b} \notin \mathbb{Z}$. Therefore, $\frac{1}{b}$ has a terminating decimal expansion.

QED

Now consider $\frac{1}{3}$ in base 10. When 1 is divided by 3, it results in a nonterminating decimal expansion that has a repetend with a period of 1. This is because 3 is a factor of $9 = 10 - 1$. This is also true for $\frac{1}{7}$ in base 8, base 15, and so on. This pattern leads to Theorem 9. The proof goes as follows:

**Theorem 9.** Let $b$ be prime. Let $\gamma = \beta - 1$. If $\gamma$ is a multiple of $b$, then the period of the repetend of $\frac{1}{b}$ has a period of 1.

**Proof.**
Let $b$ be prime. Consider base $\beta$. Let $\gamma = \beta - 1$. Let $\gamma$ be a multiple of $b$. Then $\gamma = bh$ for some $h \in \mathbb{Z}$. Let $\frac{1}{b}$ have a repetend with a period of $k$. Then, by Lemma 2, $(\beta^k - 1)\frac{1}{b} \in \mathbb{Z}$. Consider

$$(\beta^k - 1)\frac{1}{b} = \beta \gamma \gamma \ldots \gamma \frac{1}{b} = \underbrace{111 \ldots 1(\gamma)(\frac{1}{b})}_k = \underbrace{111 \ldots 1(\gamma)(\frac{1}{b})}_k = \underbrace{111 \ldots 1(h)}_k = \underbrace{hhh \ldots h}_k.$$  

$(\beta^k - 1)\frac{1}{b}$ is the whole number representation of the repetend of $\frac{1}{b}$. Thus, $\frac{1}{b}$ has a repetend whose period is 1.

QED

Theorem 8 and Theorem 9 give us necessary conditions for the remaining theorems. Before Theorem 10 can be proven, Lemma 4 is needed. The proof of Lemma 4 goes as follows:

**Lemma 4.** Let $b$ be a prime. If the period of the repetend of $\frac{1}{b}$ is $k$ in base $\beta$, then $b \nmid \underbrace{111 \ldots 1}_k$ if $t < k$.

**Proof.**
Let $b$ be prime. Let the period of the repetend of $\frac{1}{b}$ be $k$ in base $\beta$. Let $b$ be prime. Consider base $\beta$. Let $\gamma = \beta - 1$. Let $\gamma$ not be a factor of $\gamma$. Let the period of the repetend of $\frac{1}{b}$ be $k$. By Lemma 2, $(\beta^k - 1)\frac{1}{b} \in \mathbb{Z}$. Let $r = (\beta^k - 1)\frac{1}{b}$. Then, $rb = \beta^k - 1$. Thus, $b|(\beta^k - 1)$. Consider

$$\beta^k - 1 = \underbrace{\gamma \gamma \ldots \gamma}_k = (\gamma)(\underbrace{111 \ldots 1}_k).$$  

So, $b|\underbrace{111 \ldots 1}_k$. $b \nmid \gamma$ and $b$ is prime so $b|\underbrace{111 \ldots 1}_k$. Suppose towards a contradiction that $b|\underbrace{111 \ldots 1}_k$, for some $t \in \mathbb{N}$ such that $t < k$. Then, $b|\underbrace{111 \ldots 1}_k$. But, $\gamma(\underbrace{111 \ldots 1}_t) = \beta^t - 1$. So, $b|(\beta^t - 1)$ Then, $\exists u \in \mathbb{Z}$ such that $bu = \beta^t - 1$. Then,

$$bu = \beta^t - 1 \implies u = (\beta^t - 1)\frac{1}{b} = (\beta^t)\frac{1}{b} - \frac{1}{b}.$$  

Then $u$ is the repetend of $\frac{1}{b}$ expressed as a whole number, and $\frac{1}{b}$ has a repetend that has a period of no more than $t$. This is a contradiction because $t < k$. Thus, $b \nmid \underbrace{111 \ldots 1}_t$ where $t < b - 1$.

QED
Using this result, Theorem 10 can now be proven:

**Theorem 10.** Let \( b \) be prime. Let \( b \) not be a factor of \( \beta \) or \( \gamma = \beta - 1 \). Then the sum of the digits of the repetend of \( \frac{1}{b} \) in base \( \beta \) is divisible by \( \gamma \).

**Proof.**
Let \( b \) be prime with \( b \) not be a factor of \( \beta \) or \( \gamma = \beta - 1 \). Consider \( \frac{1}{b} \) in base \( \beta \). Let the period of the repetend of \( \frac{1}{b} \) be \( k \). By Lemma 2, we know that \( \beta^k \left( \frac{1}{b} \right) - \left( \frac{1}{b} \right) \) is an integer.

\[
\beta^k \left( \frac{1}{b} \right) - \left( \frac{1}{b} \right) = \left( \frac{1}{b} \right) \gamma \gamma \ldots \gamma \left( \frac{1}{b} \right) = \gamma \left( \frac{1}{b} \right) \times \frac{1}{b} \left( \frac{1}{b} \right) = \gamma \left( \frac{1}{b} \right) \times 1 = \gamma \beta^{-1}.
\]

By Lemma 4, \( b \mid 111\ldots1 \) so \( \left( \frac{1}{b} \right) \in \mathbb{Z} \). Thus, \( \gamma \beta^k \left( \frac{1}{b} \right) - \left( \frac{1}{b} \right) \).  

**QED**

Recall *Property One* of the circle diagrams of full period primes. This property states that digits of the repetend that are located diametrically from each other have a sum of 1. If the period is odd, then this property simply cannot be true because the digits of the repetend cannot line up across from each other as when the period of the repetend is even (e.g. the circle diagram for \( \frac{1}{7} \)). This pattern leads to Theorem 11 which further generalizes Theorem 4:

**Theorem 11.** Let \( b \) be prime with \( b \) not a factor of \( \beta \) or \( \gamma \). If the period of \( \frac{1}{b} \) is even in base \( \beta \), then \( d_1 + d_{k+1} = d_2 + d_{k+2} = \ldots = d_k + d_{2k} = \gamma \).

**Proof.**
Let \( b \) be prime with \( b \) not a factor of \( \beta \) or \( \gamma = \beta - 1 \). Let the period of \( \frac{1}{b} \) be \( t \) and be even. So, \( \exists k \in \mathbb{Z} \) such that \( t = 2k \). By Lemma 2, \( \beta^{2k} - 1 \left( \frac{1}{b} \right) \in \mathbb{Z} \). By Lemma 4, \( b \mid 111\ldots1 \) but \( b \not\mid 111\ldots1 \). Consider \( \left( \beta^k + 1 \right) \left( \frac{1}{b} \right) \).

\[
\left( \beta^k + 1 \right) \left( \frac{1}{b} \right) = \beta^k \left( \frac{1}{b} \right) + 1 \left( \frac{1}{b} \right) = \beta^k \left( \frac{1}{b} \right) \times \frac{1}{b} \left( \frac{1}{b} \right) = \beta^k \gamma \gamma \ldots \gamma \left( \frac{1}{b} \right) = \gamma \beta^k \left( \frac{1}{b} \right).
\]

So \( b \left( \beta^k + 1 \right) (111\ldots1) \). Since \( b \) is prime and \( b \not\mid (111\ldots1) \), \( b \mid \beta^k + 1 \). So, \( \exists g \in \mathbb{Z} \) such that \( bg = \beta^k + 1 \). Now, \( g = (\beta^k + 1) \frac{1}{b} \).

\[
\frac{1}{b} = d_1 d_2 \ldots d_k \ldots d_{k+1} d_{k+2} \ldots d_{2k} \ldots
\]

\[
\beta^k \frac{1}{b} + \frac{1}{b} = d_1 d_2 \ldots d_k d_{k+1} d_{k+2} \ldots d_{2k} \ldots + d_1 d_2 \ldots d_{2k} \ldots
\]

Because \( \left( \beta^k + 1 \right) \frac{1}{b} \in \mathbb{Z} \) and both terms of the sum are nonterminating decimals in base \( \beta \), \( d_{k+1} + d_1 = d_{k+2} + d_2 = \ldots = d_{2k} + d_k = \gamma \).

**QED**

The final theorem involves how different summations behave when looking at the numerators of the circle diagrams. Consider Figure 1. It is known that numerators that are located diametrically from each other have a sum that is divisible by 7 as proven in Theorem 6, but there is another pattern that can be observed. The period of the repetend of \( \frac{1}{b} \) is 6. \( \frac{1}{b} \) is divisible by 1, 2, 3, and 6. In the case of \( \frac{1}{7} \), Theorem 6 can be reworded to state that the sum of every 3rd numerator has a sum divisible by 7. It also true that the sum of every other numerator and the sum of every other numerator (or every 2nd) is divisible by 7; however, when going around the circle once, it is not true that every 6th remainder is divisible by 7. This pattern is true for all circle diagrams that can be generated. This is summarized in Theorem 12:

**Theorem 12.** Let \( b \) be a prime. Consider base \( \beta \). Let \( \gamma = \beta - 1 \). Let \( b \) not be a factor of \( \beta \) or \( \gamma \). Let \( k \) be the period of the repetend of \( \frac{1}{b} \). Let \( m < k \) with \( mt = k \) for some \( t \in \mathbb{Z} \). Let \( a \in \mathbb{Z} \) with \( 1 \leq a \leq m \). Let \( r_j \in \langle \beta \rangle \) with \( 1 \leq j \leq k \). Then \( b! \sum_{n=0}^{t-1} r_{mn+a} \).

**Proof.**
Let \( b \) be a prime. Consider base \( \beta \). Let \( \gamma = \beta - 1 \). Let \( b \) not be a factor of \( \beta \) or \( \gamma \). Let \( k \) be the period of the repetend of \( \frac{1}{b} \). Consider \( \langle \beta \rangle \). Since the period of the repetend of \( \frac{1}{b} \) is \( k \), \( |\langle \beta \rangle| = k \). Let \( m < k \) with \( mt = k \) for
some \( t \in \mathbb{Z} \). By The Division Algorithm, \( \forall \alpha \in \mathbb{Z}, \exists v, a \in \mathbb{Z} \) such that \( \alpha = mv + a \) with \( 1 \leq a \leq m \). Recall the recursive way to find remainders. So, \( r_j \equiv \beta^{j-1} \pmod{b} \) with \( 1 \leq j \leq k \). Consider \( \sum_{n=0}^{t-1} r_{mn+a} \).

\[
\sum_{n=0}^{t-1} r_{mn+a} \equiv \sum_{n=0}^{t-1} \beta^{mn+a-1} \pmod{b}.
\]

Let \( S = \sum_{n=0}^{t-1} \beta^{mn+a-1} \). Then

\[
\beta^m S \equiv S \pmod{b} \implies \beta^m S - S \equiv 0 \pmod{b} \implies (\beta^m - 1)S \equiv 0 \pmod{b}.
\]

Consider

\[
\beta^m - 1 = \underbrace{\gamma \gamma \ldots \gamma}_{m} = \gamma(111\ldots1).
\]

\( b \nmid \gamma \) since \( b \) is not a factor of \( \gamma \), and \( b \nmid 111\ldots1 \) by Lemma 5, so \( \beta^m - 1 \not\equiv 0 \pmod{b} \). Since \( b \) is prime and \( \beta^m - 1 \not\equiv 0 \pmod{b} \), \( S \equiv 0 \pmod{b} \). Since \( S \equiv 0 \pmod{b} \), \( b \mid \sum_{n=0}^{t-1} r_{mn+a} \).

\[QED\]

A question for further research is whether or not there is a way to determine which primes are going to be full period primes in a given base. For example, 7, 17, and 19 are full period primes in base 10, but 11 and 13 are not. Perhaps an algorithm could be developed to find which primes are going to be full period primes, or maybe something can be proven to show why some primes are not full period primes.

REFERENCES

