Divisibility of the Determinant of a Class of Matrices with Integer Entries

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Abstract

Let \( x_i = \sum_{j=1}^{n} a_{ij}10^{n-j} \), with \( i = 1, \ldots, n \), and with each \( a_{ij} \in \mathbb{Z} \) be given. Let \( A = [a_{ij}] \), and suppose that \( k \in \mathbb{Z} \) is given. If each \( x_i \) is divisible by \( k \), then we show that \( \det(A) \) is also divisible by \( k \).

1 Introduction

The following is proposed as number 1175 in the Problem Department of [1].

It is known that the numbers 14529, 15197, 20541, 38911, and 59619 are multiples of 167. Without actually calculating, prove that the determinant of the 5-by-5 matrix \( A \) is also a multiple of 167, where

\[
A = \begin{bmatrix}
1 & 4 & 5 & 2 & 9 \\
1 & 5 & 1 & 9 & 7 \\
2 & 0 & 5 & 4 & 1 \\
3 & 8 & 9 & 1 & 1 \\
5 & 9 & 6 & 1 & 9 \\
\end{bmatrix}.
\]

In this problem we are given a 5-by-5 matrix and calculating its determinant by hand is tedious. We look at a similar problem, but with smaller

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dimension. First, we look at examples involving 2-by-2 matrices to see if this case holds: if we are given two two-digit integers (say, $x = a_1a_2$ and $y = b_1b_2$) and if these integers are divisible by a given integer $k$, is it true that $k$ also divides the determinant of the matrix \[
abla a_1 \ \ a_2 \\
\ b_1 \ \ b_2\]

We investigate this case and see if it extends to higher dimensions, that is to the 3-by-3 case, to the 4-by-4 case, and in general, to the $n$-by-$n$ case. Although the original problem gave a very specific matrix, we solve a generalization of this problem and also look at other problems regarding the determinant of the matrix.

We let $\mathbb{Z}$ be the set of all integers, we let $\mathbb{N}$ be the set of all natural numbers, and we let $M_n(\mathbb{Z})$ be the set of all $n$-by-$n$ matrix with integer entries. We also denote the determinant of a matrix $A$ by $\det(A)$.

A two digit natural number $ab$ is really $a(10) + b$, and a three digit natural number $abc$ is actually $a(10^2) + b(10^1) + c(10^0)$. In general, an $n$ digit natural number $a_1a_2a_3a_4\cdots a_n$ actually means

$$a_1(10^{n-1}) + a_2(10^{n-2}) + a_3(10^{n-3}) + \cdots + a_{n-1}(10^1) + a_n(10^0)$$

where $0 \leq a_i \leq 9$ for each $i = 1, 2, \ldots, n$, and $a_1 \neq 0$.

**Example 1** The numbers 72 and 18 are divisible by 3. The matrix

$$\begin{bmatrix} 7 & 2 \\ 1 & 8 \end{bmatrix}$$

has determinant $7(8) - 2(1) = 54$, which is also divisible by 3.

**Example 2** The numbers 16 and 36 are divisible by 4. The matrix

$$\begin{bmatrix} 1 & 6 \\ 3 & 6 \end{bmatrix}$$

has determinant $1(6) - 6(3) = -12$, which is divisible by 4 as well.

In these examples, we see that if two two-digit positive integers $a_1a_2$ and $b_1b_2$ are multiples of a positive integer $k$, then the determinant of the matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ (which is $a_1b_2 - a_2b_1$), is also divisible by $k$. Note that even when the determinant of $A$ is 0, then $k$ also divides the determinant of $A$. 
Proposition 3  Let $e, f \in \mathbb{N}$ be two digit integers, say $e = 10a + b$ and $f = 10c + d$, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $k$ is a positive integer that divides both $e$ and $f$, then $k$ divides $\det(A)$.

Proof. We are given the two equations

$10a + b = e \quad (1)$

and

$10c + d = f. \quad (2)$

Multiplying equation (1) by $-c$ and equation (2) by $a$ and adding these equations we get,

$ad - bc = af - ce.$

Now, $e = kr$ and $f = kg$, where $r$ and $g$ are integers. Hence, $ad - bc = a(kg) - c(kr) = k(ag - cr)$, so that $k$ divides $(ad - bc)$.

Therefore, $k$ divides $\det(A)$. \qed

Now, we take a look at a particular 3-by-3 example.

Example 4 The numbers 156, 228, and 276 are multiples of 12. A calculation shows that the determinant of the matrix $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 2 & 8 \\ 2 & 7 & 6 \end{bmatrix}$ is 36, which is also a multiple of 12.

We now take a closer look at the case $n = 3$. Let

$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad (3)$

and notice that

$\det(A) = aei - afh + bfg - bdi + cdh - ceg. \quad (4)$

Suppose $m, n,$ and $p$ are three digit integers, say $m = 100a + 10b + c,$ $n = 100d + 10e + f,$ and $p = 100g + 10h + i$. Assume that $m, n,$ and $p$ are divisible by $k$. We show that $k$ divides $\det(A)$.
We have following system of equations

\[100a + 10b + c = m \quad (5)\]
\[100d + 10e + f = n \quad (6)\]
\[100g + 10h + i = p. \quad (7)\]

If \(a = d = g = 0\), then \(\det(A) = 0\), and is divisible by \(k\). Hence, we may assume that one of \(a, d,\) or \(g\) is nonzero. Without loss of generality, suppose \(d \neq 0\).

Multiplying (5) by \(d\) on both sides, multiplying (6) by \(-a\), and adding these equations, we get

\[10 (bd - ea) + cd - fa = md - na \quad (8)\]

Similarly if multiply (6) by \(g\) and if we multiply (7) by \(-d\) and adding these equations, we get

\[10 (ge - hd) + gf - id = gn - pd \quad (9)\]

Multiply (8) by \(-(ge - hd)\), multiply (9) by \((bd - ea)\) and add these two equations: \((bd - ea)(gf - id) - (cd - fa)(ge - hd) = (bd - ea)(gn - pd) - (md - na)(ge - hd)\). Simplifying the left hand side, we get

\[d (aei - afh + bg - bdi + cdh - ceg),\]

while on the right hand side, we have

\[d(dhm - egm - ahh + bdp - bgn - aep).\]

Since \(d \neq 0\), we can divide by \(d\) to get

\[aei - afh + bg - bdi + cdh - ceg = dhm - egm - ahh + bdp - bgn - aep. \quad (10)\]

The left hand side of this equation is \(\det A\). Since \(m, n,\) and \(p\) are divisible by \(k\), then each of the summands on the right hand side is divisible by \(k\). We conclude that \(k\) divides \(\det A\).

Notice that although our method works, it can get very tedious when the size of the matrix becomes large. This necessitates looking for a different approach.
Before we look at the general case of this problem we define some important terminology and facts about matrices.

Let \( A = [a_{ij}] \) be an \( n \times n \) matrix and let \( M_{ij} \) denote the \((n-1)\times(n-1)\) matrix obtained from \( A \) by deleting the row and column containing \( a_{ij} \). The determinant of \( M_{ij} \) is called the minor of \( a_{ij} \). The cofactor of \( a_{ij} \) is

\[
A_{ij} = (-1)^{i+j} \det(M_{ij}).
\]

The adjoint of \( A \) is

\[
\text{adj}(A) = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}.
\]

**Example 5** Suppose \( A = \begin{bmatrix} 3 & 8 & 7 \\ 2 & 6 & 0 \\ 5 & 4 & 3 \end{bmatrix} \).

1. Then \( A_{11} = \begin{vmatrix} 6 & 0 \\ 3 & 4 \end{vmatrix} = 18, \ A_{12} = -\begin{vmatrix} 2 & 0 \\ 5 & 3 \end{vmatrix} = -6, \ldots \) and \( A_{33} = \begin{vmatrix} 3 & 8 \\ 2 & 6 \end{vmatrix} = 2, \)

2. \( \text{adj}(A) = \begin{bmatrix} 18 & -6 & -22 \\ 4 & -26 & 28 \\ -42 & 14 & 2 \end{bmatrix}, \) and

3. \( \det(A) = 3(18) - 8(6) + 7(-22) = -148. \)

If \( A = [a_{ij}] \in M_n(\mathbb{Z}) \) then \( \text{adj}(A) \in M_n(\mathbb{Z}) \) because \( \text{adj}(A) \) is calculated by adding, subtracting and multiplying the entries of \( A \). The following is a connection between a nonsingular matrix and its adjoint [2, page 116].

**Lemma 6** Let \( A \) be an \( n \times n \) nonsingular matrix with complex entries. Then \( A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \).

Now, we are ready to solve our problem.
Theorem 7 Let $A = [a_{ij}] \in M_n(\mathbb{Z})$ and fix a nonzero $k \in \mathbb{Z}$. Set $b = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}^T$ where $b_i = a_{i1}(10^{n-1}) + a_{i2}(10^{n-2}) + \cdots + a_{in}(10^0)$. If $b_1, b_2, \ldots, b_n$ are all divisible by $k$, then $\det(A)$ is also divisible by $k$.

Proof. Let $A = [a_{ij}] \in M_n(\mathbb{Z})$ be given. First, notice that if $A$ is singular, then $\det(A) = 0$ is divisible by $k$. Hence, we may assume that $A$ is nonsingular.

Suppose $A$ is nonsingular and set $z = \begin{bmatrix} 10^{n-1} & \cdots & 10^0 \end{bmatrix}^T$. Then we have $Az = b$. We are given that for each $i = 1, \ldots, n$, we have $b_i = kr_i$ for some $r_i \in \mathbb{Z}$. Set $r = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}^T$. Then $b = kr$.

Now, $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$. Hence
\[
z = A^{-1}b = \frac{1}{\det(A)} \text{adj}(A)b = \frac{1}{\det(A)} \text{adj}(A)kr
\]
so that
\[
\frac{\det(A)}{k}z = \text{adj}(A)r
\]
Since $\text{adj}(A)r \in M_n(\mathbb{Z})$, then $\frac{\det(A)}{k} \in \mathbb{Z}$. Thus, $\det(A)$ is divisible by $k$. ■

A natural question to ask is: for $A \in M_n(\mathbb{Z})$, when is $A^{-1} \in M_n(\mathbb{Z})$?

Theorem 8 Let $A \in M_n(\mathbb{Z})$ be nonsingular. Then $A^{-1} \in M_n(\mathbb{Z})$ if and only if $\det(A) = \pm 1$.

Proof. Let $A \in M_n(\mathbb{Z})$ be given. Then $\det(A) \in \mathbb{Z}$ because the determinant is calculated by adding, subtracting and multiplying entries of a matrix.

If $A^{-1} \in M_n(\mathbb{Z})$, then $\det(A^{-1}) \in \mathbb{Z}$. But $\det(A^{-1}) = \frac{1}{\det(A)}$. Therefore, $\frac{1}{\det(A)}$ is also an integer. Hence $\det(A) = \pm 1$.

Conversely, if $\det(A) = \pm 1$, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \in M_n(\mathbb{Z})$ since $\text{adj}(A) \in M_n(\mathbb{Z})$. ■

References
