THE NUMBER OF ADMISSIBLE SEQUENCES FOR INDECOMPOSABLE SERIAL RINGS WITH A SIMPLE PROJECTIVE MODULE

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ABSTRACT: We prove that the number of admissible sequences for indecomposable serial rings with a simple projective module and with $n$ basic idempotents ($n \geq 1$) is the $n-1^{st}$ Catalan number.

INTRODUCTION

Throughout this paper, $R$ will denote a basic indecomposable left Artinian ring (with unity) with $\{e_1, \ldots, e_n\}$ a basic set of primitive idempotents and with $J$ the Jacobson radical of $R$. The collection $\{Re_1/Je_1, Re_2/Je_2, \cdots, Re_n/Je_n\}$ forms a complete set of pairwise non-isomorphic simple left $R$-modules. Furthermore, $\{Re_1, Re_2, \cdots, Re_n\}$ forms a complete set of pairwise non-isomorphic indecomposable projective left $R$-modules.

The ring $R$ is serial if each of the indecomposable projective $R$-modules (both left and right) has a unique composition series. Denoting the composition lengths of the indecomposable projective left $R$-modules by $c_i = c(Re_i)$, then the sequence
$c_1, c_2, \ldots, c_n$ is called an *admissible sequence* for $R$. The admissible sequence for a serial ring satisfies the following inequalities (see [AF]):

$$2 \leq c_i \leq c_{i-1} + 1 \text{ for } i = 2, \ldots, n$$

$$c_1 \leq c_n + 1$$

Throughout the remainder of this paper, we assume that $R$ is an indecomposable serial ring with a simple projective left $R$-module. The admissible sequence can be reordered so that $c_1 = 1$. We then have that

$$c_1 = 1 \text{ and } 2 \leq c_i \leq c_{i-1} + 1 \text{ for } i = 2, \ldots, n \quad (1.1)$$

We denote by $a_n$ the number of admissible sequences of length $n$ ($n \geq 1$).

**EXAMPLE:** We calculate $a_n$ for $1 \leq n \leq 6$ by listing all admissible sequences of length $n$. (Note that we list the sequence $c_1, c_2, \ldots, c_n$ as $c_1c_2\cdots c_n$).

$$a_1 = 1$$

1

$$a_2 = 1$$

12

$$a_3 = 2$$

122, 123

$$a_4 = 5$$

1222, 1223, 1232, 1233, 1234
\[ a_5 = 14 \]
\[ 12222,12223,12232,12233,12234,12322,12323,12332,12333,12334,12342,12343,12344,12345 \]

\[ a_6 = 42 \]
\[ 122222,122223,122232,122233,122234,122322,122323,122332,122333,122334,122342,122343,122344,122345,123222,123223,123232,123233,123234,123322,123323,123332,123333,123334,123342,123343,123344,123345,123422,123423,123432,123433,123434,123442,123443,123444,123445,123452,123453,123454,123455,123456 \]

**PROOFS**

The \( n^{th} \) Catalan number, \( b_n \), is defined as

\[ b_n = \frac{1}{n+1} \binom{2n}{n} \text{ for } n \geq 0 \]

**THEOREM** : The number of admissible sequences for indecomposable serial rings with a simple projective module and with \( n \) basic idempotents (\( n \geq 1 \)) is the \( n - 1^{st} \) Catalan number.

Note that the theorem is established for \( 1 \leq n \leq 6 \) in the preceding example. The proof of the theorem requires the following combinatorial identities (see [R]).

\[ \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r} \text{ for } n \geq r \geq 1 \quad (1.2) \]

\[ \binom{n}{r} = \sum_{k=0}^{r} \binom{n-1-k}{r-k} \text{ for } n > r \geq 1 \quad (1.3) \]
\[
\binom{2n+1}{n} = \binom{2n+1}{n+1} \quad \text{for } n \geq 0 \quad (1.4)
\]

\[
b_n = \binom{2n}{n} - \binom{2n}{n-1} \quad \text{for } n \geq 1 \quad (1.5)
\]

We denote by \(a_{n,m}\) the number of admissible sequences \(c_1, c_2, \cdots, c_n\) with \(c_n = m\). By repeated application of (1.1), \(c_n \leq c_{n-1} + 1 \leq c_{n-2} + 2 \leq \cdots \leq c_1 + (n-1) = n\). Thus the sequence 1, 2, \cdots, \(n-1\), \(n\) is the only admissible sequence of length \(n\) with \(c_n = n\). Therefore \(a_{n,n} = 1\) and

\[
a_n = \sum_{m=2}^{n} a_{n,m} \quad \text{for } n \geq 2
\]

**LEMMA :**

\[
a_{n,m} = \binom{2n-m-2}{n-m} - \binom{2n-m-2}{n-m-1} \quad \text{for } n \geq 4 \text{ and } 2 \leq m < n
\]

**Proof of Lemma :** We induct on \(n\). The case \(n = 4\) is easily verified in the preceding example. Let \(n > 4\). Let \(c_1, c_2, \cdots, c_{n-1}\) be an admissible sequence of length \(n-1\). Then this sequence can be extended to an admissible sequence of length \(n\) by taking \(c_n\) to be any integer such that \(2 \leq c_n \leq c_{n-1} + 1\). Conversely, if \(c_1, c_2, \cdots, c_n\) is an admissible sequence of length \(n\), then clearly \(c_1, c_2, \cdots, c_{n-1}\) is also an admissible sequence.

Thus, every admissible sequence of length \(n\) with \(c_n = m\) can be obtained by extending an admissible sequence of length \(n-1\) with \(c_{n-1}\) equal to one of the values \(m-1, m, \cdots, n-1\). Therefore we have that

\[
a_{n,m} = \sum_{j=m-1}^{n-1} a_{n-1,j} = a_{n-1,n-1} + \sum_{j=m-1}^{n-2} a_{n-1,j}
\]
= 1 + \sum_{j=m-1}^{n-2} \left[ \binom{2n-j-4}{n-j-1} - \binom{2n-j-4}{n-j-2} \right] \quad \text{(induction hypothesis)}

= \sum_{j=m-1}^{n-1} \binom{2n-j-4}{n-j-1} - \sum_{j=m-1}^{n-2} \binom{2n-j-4}{n-j-2}

= \sum_{k=0}^{n-m} \binom{2n-m-3-k}{n-m-k} - \sum_{k=0}^{n-m-1} \binom{2n-m-3-k}{n-m-1-k} \quad \text{(let } k = j-m+1 \text{)}

= \binom{2n-m-2}{n-m} - \binom{2n-m-2}{n-m-1} \quad \text{(by 1.3)}

\textbf{Proof of Theorem:} \text{ Let } n \geq 4. \text{ Thus }

a_n = a_{n,n} + \sum_{m=2}^{n-1} a_{n,m}

= 1 + \sum_{m=2}^{n-1} \left[ \binom{2n-m-2}{n-m} - \binom{2n-m-2}{n-m-1} \right] \quad \text{(by Lemma)}

= \sum_{m=2}^{n} \binom{2n-m-2}{n-m} - \sum_{m=2}^{n-1} \binom{2n-m-2}{n-m-1}

= \sum_{k=0}^{n-2} \binom{2n-4-k}{n-2-k} - \sum_{k=0}^{n-3} \binom{2n-4-k}{n-3-k} \quad \text{(let } k = m-2 \text{)}

= \binom{2n-3}{n-2} - \binom{2n-3}{n-3} \quad \text{(by 1.3)}

= \binom{2n-3}{n-1} - \binom{2n-3}{n-3} \quad \text{(by 1.4)}

= \left[ \binom{2n-3}{n-1} + \binom{2n-3}{n-2} \right] - \left[ \binom{2n-3}{n-2} + \binom{2n-3}{n-3} \right]

= \binom{2n-2}{n-1} - \binom{2n-2}{n-2} \quad \text{(by 1.2)}
\[ = b_{n-1} \]  
(by 1.5)

REMARK: The referee has noted that the admissible sequences of length \( n \) can be shown to be in one-to-one correspondence with the plane rooted trees with \( n - 1 \) edges \( (n \geq 2) \). Furthermore, as was noted by Eggleton and Guy (see [EG]), the number of such trees is the \( (n - 1)^{st} \) Catalan number.

REFERENCES

