DESIGNING ROLLER COASTERS

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Abstract. Under the assumption that the only external force acting on a particle is gravity, we determine the banking necessary to keep the particle on the track of a roller coaster.

Introduction. Mankind has always been fascinated with motion, the faster, the better. Even more than five centuries ago, ways had been discovered to create that motion by harnessing one of the most omnipresent forces in nature, gravity. Beginning in the 15th and 16th centuries, Russian nobles in St. Petersburg would entertain themselves with the first known gravity rides called “Russian Mountains”. These were huge blocks of ice that were fashioned into sleds, with straw or fur placed on the icy seat for passenger comfort. Sand was used to help slow the sled down at the end of the ride to keep it from crashing. Later, more elaborate wooden sleds were built with iron runners to increase the speed and the intensity of the ride. The addition of wheels did not come until the mid-seventeenth hundreds.

Rides of this type did not come west for quite some time. A Russian Mountain was built in the Ternes Quarter in Paris in 1804 and proved very popular there. In 1817, a coaster called “The Aerial Walk” was constructed in Paris. This was the world’s first racing coaster. Coasters came to the United States soon afterwards. The introduction of amusement parks in the United States is what literally got the roller coasters off the ground. The first amusement park in American history was located on Coney Island in New York, built in 1875. Railway companies, in search of ways to keep passenger usage up on the weekends, set up parks at the end of the rail lines and introduced weekend and summer activities. The first rides were carousels, but in 1884, the first true roller coaster; a gravity switchback train created by La Marcus Thomson, was introduced to the park at Coney Island. For many years, roller coasters remained for the most part, low-speed wooden contraptions with little vertical displacement. However in 1901, Coney Island’s “Flip Flap Railway” became the first coaster to have a vertical loop. Unfortunately, riders kept passing out, so the ride was closed. Shortly thereafter, the inventor Edward Prescott solved the problem by introducing an elliptical loop. This reduces the g forces on the rider upon entering and exiting the loop.

A new era for roller coaster design began with the opening of Disneyland in 1955. In 1959 Disney introduced the “Matterhorn”, the first tubular steel coaster. The fact that coasters were now made out of steel instead of wood led to the exciting features we expect from today’s coasters; loops, corkscrews, and high speeds. The race to build the ultimate roller coaster took off in the late 1980’s. The “Magnum XL-200” coaster, which opened in 1989 at the Cedar Point amusement park in Ohio, was the first to exceed 200 ft. in height and in 1996 “Superman-The Escape” located in California was the first to break the 100 mph mark. There are rumors that Arrow Dynamics is obtaining approval for a 350 ft, 100 mph coaster that will feature the world’s tallest loop. One can only imagine what heights and speeds the roller coasters of the future will achieve. For more history of the roller coaster see [4,5].
Analysis of the problem. Although roller coasters can be very exciting, they can also be very dangerous. In the 1980's there were approximately 5 deaths per year due to coasters. This has decreased in recent years due to advances in design software. Computer modeling plays a crucial role throughout the design process. Speed and g forces are relatively easy to calculate, but what sets coaster companies apart from each other are the different banking programs to calculate the turns and make the ride smooth. The purpose of this paper is to design the correct banking of a roller coaster track so that a particle moving in space will stay on a given curve in space. For simplicity, we assume that the only external force acting on the particle is due to gravity and that the track is frictionless.

Let \( C \) be a smooth curve in space with parameterization \( r(\tau) = (x(\tau), y(\tau), z(\tau)) \) for \( \alpha \leq \tau \leq \beta \). Recall that the curvature of \( C \) can be written as \( \kappa(\tau) = \frac{|r'(\tau) \times r''(\tau)|}{|r'(\tau)|^3} \). We will define \( T(\tau) := \frac{r'(\tau)}{|r'(\tau)|} \) to be the unit tangent vector and \( N := \frac{r'(\tau) \times r''(\tau)}{|r'(\tau) \times r''(\tau)|} \) to be the principal unit normal to \( C \) at \( r(\tau) \). Then \( B(\tau) := T(\tau) \times N(\tau) \) is the binormal to \( C \) at \( r(\tau) \). Suppose our particle which represents the roller coaster has initial position \( r(\alpha) \) and initial velocity \( S_0T(\alpha) \).

Let \( R(t) = (X(t), Y(t), Z(t)) \) denote the position of the roller coaster \( t \) seconds after entering the track. Also, let \( V(t) \) be the velocity of the roller coaster, \( A(t) \) its acceleration, and \( \tau = \tau(t) \) the value of \( \tau \) for which \( R(t) = r(\tau) \). Thus, \( R(0) = r(\alpha) \) and \( V(t) = S(t)T(\tau) \), where \( S(t) \) is the speed of the particle at time \( t \). If we let \( \theta(\tau) \) be the angle of inclination from the track laid across the curve \( C \) at \( r(\tau) \), then the track has direction

\[
U(\tau) = \cos \theta(\tau)N(\tau) - \sin \theta(\tau)B(\tau).
\]

As usual, we can decompose the acceleration of the roller coaster in terms of the tangential and normal components of acceleration. That is,

\[
A(t) = a_T(t)T(\tau) + a_N(t)N(\tau),
\]

where \( a_T(t) = S'(t) \) and \( a_N(t) = \kappa(\tau)S(t)^2 \). Furthermore, if we use \( \{T(\tau), N(\tau), B(\tau)\} \) as an orthonormal basis we can resolve the acceleration \( G = (0, 0, -g) \) due to gravity at the position \( r(\tau) \) as

\[
G = G \cdot T(\tau)T(\tau) + G \cdot N(\tau)N(\tau) + G \cdot B(\tau)B(\tau).
\]
Newton’s law implies that $ma_T(t)$ is equal to the component of the gravitational force in the tangential direction, $m \mathbf{G} \cdot \mathbf{T}(\tau)$, where $m$ denotes the mass of the particle. Thus

$$S'(t) = a_T(t) = \mathbf{G} \cdot \mathbf{T}(\tau) = \mathbf{G} \cdot \frac{\mathbf{V}(t)}{S(t)} = -g \frac{Z'(t)}{S(t)}.$$ 

If we integrate with respect to $t$ we obtain

$$S(t)^2 = S_0^2 - 2g(Z(t) - Z(0)) = S_0^2 - 2g(z(\tau) - z(\alpha)),$$

where $Z(0) - z(\alpha)$ is the initial height of the roller coaster. The components of the forces $m \mathbf{A}(t)$ and $m \mathbf{G}$ in the direction perpendicular to the track are compensated by the track itself, thus it only remains for us to consider the components of the acting forces in the direction of $\mathbf{U}(\tau)$. If we wish for the roller coaster to stay on the track the following condition needs to be satisfied:

$$m \mathbf{A}(t) \cdot \mathbf{U}(\tau) = m \mathbf{G} \cdot \mathbf{U}(\tau).$$

If we put equations (1), (2), and (3) into equation (4) we get

$$\kappa(\tau) S(t)^2 \cos \theta(\tau) = \mathbf{G} \cdot \mathbf{N}(\tau) \cos \theta(\tau) - \mathbf{G} \cdot \mathbf{B}(\tau) \sin \theta(\tau).$$

This equation can be written as

$$(\kappa(\tau) S(t)^2 - \mathbf{G} \cdot \mathbf{N}(\tau)) \cos \theta(\tau) + \mathbf{G} \cdot \mathbf{B}(\tau) \sin \theta(\tau) = 0.$$ 

We can write the above equation equivalently in vector form as

$$(\kappa(\tau) S(t)^2 - \mathbf{G} \cdot \mathbf{N}(\tau), \mathbf{G} \cdot \mathbf{B}(\tau)) \cdot (\cos \theta(\tau), \sin \theta(\tau)) = 0.$$ 

This implies that the two vectors are orthogonal and thus

$$\theta = \pm \frac{\pi}{2} + \text{arg}(\kappa(\tau) S(t)^2 - \mathbf{G} \cdot \mathbf{N}(\tau) + i \mathbf{G} \cdot \mathbf{B}(\tau)).$$

In order to decide which formula for $\theta$ is appropriate, we consider a particle moving along a straight line in the $xy$-plane so that $\mathbf{B} = (0, 0, 1)$. We would like $\theta$ to be $0$ in this situation. So we need to choose

$$\theta = \frac{\pi}{2} + \text{arg}(\kappa(\tau) S(t)^2 - \mathbf{G} \cdot \mathbf{N}(\tau) + i \mathbf{G} \cdot \mathbf{B}(\tau)).$$

**Examples.**

We wish to consider three curves in space and calculate the angle of banking $\theta(\tau)$ in terms of the initial speed of the particle. The first curve we wish to consider is simply a circle in space of radius $\rho$ tangent to the origin and tilted an angle of $\phi$ measured from the positive $z$ axis. Its equation can be represented as

$$r_1(\tau) = (\rho \sin(\tau), \rho(1 - \cos(\tau)) \sin(\phi), \rho(1 - \cos(\tau)) \cos(\phi)).$$

It is interesting to note that for this curve there is a critical initial speed below which the rollercoaster will not be inverted at the top of the loop. For the given curve this occurs at about 12.56 mph as illustrated in the figure below.
We will now illustrate the actual track needed when the initial speed is 13 mph. The line segment with the ball at the top represents the direction in which a person’s head would be pointing if they are riding the roller coaster in the traditional way.

The second curve is somewhat more complicated, but it still has constant curvature. It is a helix starting at the origin. Its equation is

\[ r_2(\tau) = (\sin(\pi \tau), \tau, (1 - \cos(\pi \tau))) \]

It turns out that the initial speed of the coaster needs to be at least 18.77 m.p.h. or it will fall from the track. First, we illustrate the banking angle when we have an initial speed of 18 mph. It is clear from the picture that this initial speed will not work.
Next we illustrate the banking angle when we have initial speeds of 20 and 32 mph. Note that the banking angle needed for this track stays closer to 90 degrees as the initial speed is increased.

A picture of the track when the initial speed is 20 mph. follows below.
Finally, we will consider a tornado-like curve whose curvature is not constant. If we start the particle at the top of the curve you may think of this as a waterslide.

\[ r_3(\tau) = (2\tau \cos(2\pi(4 - \tau)), 2\tau \sin(2\pi(4 - \tau)), 2(4 - \tau)). \]

The banking angle increases rapidly near the top and then levels off as shown in the following picture. The fact that the angle is “leveling off” is due to the increasing curvature of the generating curve which “cancells” the increasing speed.

Finally we show the track when the initial speed is 0 mph.

Our work can be extended in several ways. We could add friction or we could calculate the g forces associated with the turns. Another interesting problem would be to do the mathematical analysis behind finding the critical speed below which the coaster would fall of the helical track.
References.