Finite Differences
and
Derivatives

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Abstract: We shall review finite differences and some of their properties. We shall then investigate the similarities between finite differences and derivatives. Finally, we shall compare some identities involving finite sums and differences which seem similar to the identities in the Fundamental Theorems of Calculus which involves integrals and derivatives.
Background

According to [1] p. 512, finite differences appeared in the 17th Century. According to [2] p. 22, it was used heavily by Newton and much of the subject originated with him. A good presentation of finite differences, factorial polynomials, and summations appears in [2] p. 22–42. We refer the interested reader to these sections. Issac Newton was a master of finite differences and from this, one can understand more clearly what must have led to the development of calculus and the discovery of the Fundamental Theorems of Calculus. Perhaps, students could better understand how Calculus developed and appreciate the Fundamental Theorems of Calculus if they were exposed to finite differences, summations, and related materials.

Definition

For a (finite) sequence \( \{x_j\}_{j=1}^n \), we define the (first) (finite) difference of \( x_j, \Delta x_j \), as

\[
\Delta x_j \equiv x_{j+1} - x_j
\]

for \( 1 \leq j < n \). We use the notation \( \Delta x_j \) for \( \Delta x_j \) when there is no confusion.

We can define second finite differences by \( \Delta^2 x_j \equiv \Delta ( \Delta x_j ) \). Higher orders of finite differences can be defined inductively by \( \Delta^{n+1} x_j \equiv \Delta ( \Delta^n x_j ) \).

Example

Let \( \{x_j\}_{j=1}^5 = \{x_1, x_2, x_3, x_4, x_5\} = \{2, 5, 3, 7, -4\} \). Then \( \Delta x_1 = 3, \Delta x_2 = -2, \Delta x_3 = 4 \) and \( \Delta x_4 = -11 \).

Also, we have \( \Delta^2 x_1 = -5, \Delta^2 x_2 = 6 \) and \( \Delta^2 x_3 = -15; \Delta^3 x_1 = 11 \) and \( \Delta^3 x_2 = -21; \) and finally \( \Delta^4 x_1 = -32 \). The computations are illustrated below.

\[
\begin{array}{cccccccc}
\end{array}
\]
Example from Physics

Let us consider the distance an object falls starting from rest for \( t = 0, 1, 2, 3, 4, 5, 6 \). Distance is measured in feet and time in seconds.

Then, we have \( \{s_j\}_{j=1}^7 = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = \{0, 16, 64, 144, 256, 400, 576\} \);
and \( \Delta s_1 = 16, \Delta s_2 = 48, \Delta s_3 = 80, \Delta s_4 = 112, \Delta s_5 = 144 \) and \( \Delta s_6 = 176 \).

Also, \( \Delta^2 s_1 = \Delta^2 s_2 = \Delta^2 s_3 = \Delta^2 s_4 = \Delta^2 s_5 = 32; \Delta^3 s_1 = \Delta^3 s_2 = \Delta^3 s_3 = \Delta^3 s_4 = 0 \);
\( \Delta^4 s_1 = \Delta^4 s_2 = \Delta^4 s_3 = 0; \Delta^5 s_1 = \Delta^5 s_2 = 0; \) and \( \Delta^6 s_1 = 0 \). The computations are illustrated below.

\[
\begin{array}{cccc}
0 & 16 & 32 & 0 \\
16 & 32 & 0 & 0 \\
64 & 32 & 0 & 0 \\
80 & 0 & 0 & 0 \\
144 & 32 & 0 & 0 \\
112 & 0 & 0 & 0 \\
256 & 32 & 0 & 0 \\
144 & 0 & 0 & 0 \\
400 & 32 & 0 & 0 \\
576 & 176 & & \\
\end{array}
\]

The third column illustrates that the change in (average) velocity, i.e., acceleration, is a constant 32 ft/sec². Hence, one may draw the conclusion that velocity is increasing linearly.

Note on Higher Order Finite Differences

From the definitions, above, one can show the following:

\[ \Delta x_j = \Delta^1 x_j = x_{j+1} - x_j, \]
\[ \Delta^2 x_j = x_{j+2} - 2x_{j+1} + x_j, \]
\[ \Delta^3 x_j = x_{j+3} - 3x_{j+2} + 3x_{j+1} - x_j, \]
\[ \Delta^4 x_j = x_{j+4} - 4x_{j+3} + 6x_{j+2} - 4x_{j+1} + x_j, \]
\[ \vdots \]
\[ \Delta^n x_j = \sum_{k=0}^{n} (-1)^k C^n_k x_{j+n-k} = \sum_{k=0}^{n} (-1)^{n-k} C^n_{n-k} x_{j+k}, \]
\[ \vdots \]
Difference vs Derivative

For real constant $\alpha$, $\beta$, $\kappa$; (finite) sequences $\{u_j\}$, $\{v_j\}$; functions $u$, $v$ of $x$; positive integer $n$; and real constant $b$ such that $0 < b \neq 1$, we have the following identities:

1. $\Delta \kappa = 0 \quad D(\kappa) = 0$
2. $\Delta(\kappa u_j) = \kappa \Delta u_j \quad D(\kappa u) = \kappa D(u)$
3. $\Delta(u_j \pm v_j) = \Delta u_j \pm \Delta v_j \quad D(u \pm v) = D(u) \pm D(v)$
4. $\Delta(\alpha u_j \pm \beta v_j) = \alpha \Delta u_j \pm \beta \Delta v_j \quad D(\alpha u \pm \beta v) = \alpha D(u) \pm \beta D(v)$
5. $\Delta(u_j v_j) = u_j \Delta v_j + v_j \Delta u_j \quad D(uv) = uD(v) + vD(u)$

Interesting, we may notice the following commutative diagram:

\[
\begin{array}{ccc}
\Delta u_{j+1}v_{j+1} - u_jv_j & \xrightarrow{\text{Definition}} & u_jv_j \\
\| & \| & \| \\
v_{j+1}u_{j+1} - v_ju_j & \xrightarrow{\text{Product Rule}} & u_j \Delta v_j + v_j \Delta u_j \\
\| & \| & \| \\
\| & \| & \|
\end{array}
\]

$E$ denotes the Interchanging of $u$ and $v$

- $\Delta(u_j^n) = \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j) \quad D(u^n) = nu^{n-1}D(u)$
- $\approx nu_j^{n-1} \Delta u_j$, if $u_j \approx u_{j+1}$

where $u_j^n = (u_j)^n$, $u_j^{n-k} = (u_j)^{n-k}$, $u_{j+1}^{k-1} = (u_{j+1})^{k-1}$; and

- $\Delta(u_j^{-n}) = -\left( \sum_{k=1}^{n} u_j^{-k}u_{j+1}^{-n+k-1} \right) \cdot \Delta(u_j) \quad D(u^{-n}) = -nu^{-n-1}D(u)$
- $\approx -nu_j^{-n-1} \Delta u_j$, if $u_j \approx u_{j+1}$

where $u_j^{-n} = (u_j)^{-n}$, $u_j^{-k} = (u_j)^{-k}$, $u_{j+1}^{-n+k-1} = (u_{j+1})^{-n+k-1}$

\[\text{---3---}\]
\[ \Delta \left( \frac{u_j}{v_j} \right) = \frac{v_j \Delta u_j - u_j \Delta v_j}{v_j v_{j+1}} \quad D \left( \frac{u}{v} \right) = \frac{v D(u) - u D(v)}{v^2} \]

\[ \approx \frac{v_j \Delta u_j - u_j \Delta v_j}{v_j^2}, \text{ if } v_j \approx v_{j+1} \]

\[ \Delta (b^j) = b^j (b - 1) \quad D (b^x) = b^x \ln b \]

\[ \Delta (e^j) = e^j (e - 1) \quad D (e^x) = e^x \]

\[ \Delta (\sin j) = 2 \sin \frac{1}{2} \cos (j + \frac{1}{2}) \quad D (\sin x) = \cos x \]

\[ \Delta (\cos j) = -2 \sin \frac{1}{2} \sin (j + \frac{1}{2}) \quad D (\cos x) = -\sin x \]

\[ \Delta (\tan j) = \sin 1 \sec j \sec (j + 1) \quad D (\tan x) = \sec^2 x \]

\[ = \frac{\tan 1 \sec^2 j}{1 - \tan 1 \tan j} \]

\[ = \tan 1 (1 + \tan j \tan (j + 1)) \]

\[ \Delta (\log_b u_j) = \log_b \left( \frac{u_{j+1}}{u_j} \right) \quad D (\log_b x) = \frac{1}{x \ln b} \]

\[ \Delta (\ln u_j) = \ln \left( \frac{u_{j+1}}{u_j} \right) \quad D (\ln x) = \frac{1}{x} \]

\[ \Delta (\log_b (\alpha + j \beta)) = \log_b \left( 1 + \frac{\beta}{\alpha + j \beta} \right) \quad D (\log_b (\alpha + x \beta)) = \frac{\beta}{(\alpha + x \beta) \ln b} \]

\[ \Delta (\ln (\alpha + j \beta)) = \ln \left( 1 + \frac{\beta}{\alpha + j \beta} \right) \quad D (\ln (\alpha + x \beta)) = \frac{\beta}{\alpha + x \beta} \]

Notice the strong similarity between finite difference and differentiation.

proofs

Proofs of the derivative formulas can be found in many calculus texts, e.g., [3].

(1) Let \( x_j = x_{j+1} = \kappa \). Then \( \Delta \kappa = \Delta x = x_{j+1} - x_j = \kappa - \kappa = 0 \). \[ \square \]

(2) \( \Delta (\kappa u_j) = \kappa u_{j+1} - \kappa u_j = \kappa (u_{j+1} - u_j) = \kappa \Delta u_j \). \[ \square \]

(3) \( \Delta (u_j \pm v_j) = (u_{j+1} \pm v_{j+1}) - (u_j \pm v_j) = u_{j+1} \pm v_{j+1} - u_j \mp v_j = u_{j+1} - u_j \pm v_{j+1} \mp v_j \]

\[ = u_{j+1} - u_j \pm (v_{j+1} - v_j) = \Delta u_j \pm \Delta v_j. \]

\[ \square \]

(4) \( \Delta (\alpha u_j \pm \beta v_j) = \Delta (\alpha u_j) \pm \Delta (\beta v_j) = \alpha \Delta u_j \pm \beta \Delta v_j. \]

\[ \square \]

(5) \( \Delta (u_j v_j) = u_{j+1} v_{j+1} - u_j v_j = u_j v_{j+1} - u_j v_j + v_{j+1} u_{j+1} - v_{j+1} u_j \]

\[ = u_j (v_{j+1} - v_j) + v_{j+1} (u_{j+1} - u_j) = u_j \Delta v_j + v_{j+1} \Delta u_j. \]

\[ \square \]
(6) \( \Delta(u_j^1) = \Delta(u_j) = 1 \cdot \Delta(u_j) = (u_j^0u_{j+1}^0) \cdot \Delta(u_j) = \left( \sum_{k=1}^{1} u_j^{1-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j) \). This prove (6) for \( n = 1 \).

\( \Delta(u_j^2) = \Delta(u_j u_j) = \Delta u_j + u_{j+1} \Delta u_j = (u_j + u_{j+1}) \Delta u_j = \left( \sum_{k=1}^{2} u_j^{2-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j) \). This prove (6) for \( n = 2 \).

Assume that (6) holds for \( n \), i.e., \( \Delta(u_j^n) = \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j) \). Then,

\[
\Delta(u_j^{n+1}) = \Delta(u_j^n u_j) = u_j^n \Delta u_j + u_{j+1} \Delta(u_j^n) = u_j^n \Delta u_j + u_{j+1} \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \Delta u_j
\]

\[
= u_j^n \Delta u_j + \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \Delta u_j
\]

\[
= u_j^{n+1-1}u_{j+1}^{1-1} \Delta u_j + \left( \sum_{k=2}^{n+1} u_j^{n+1-k}u_{j+1}^{k-1} \right) \Delta u_j
\]

\[
= \left( \sum_{k=1}^{n+1} u_j^{n+1-k}u_{j+1}^{k-1} \right) \Delta u_j \quad \text{and (6) holds for } n+1. \quad \text{Hence, (6) follows by induction for } n > 0. \quad \text{We will prove the second part of (6) for } -n \text{ after (7).}
\]

\[
(7) \quad \Delta \left( \frac{u_j}{v_j} \right) = \frac{u_{j+1} - u_j}{v_j + 1} \quad \text{v}_j \quad \text{v}_j + 1 \quad \text{u}_j \quad \text{u}_j + 1 \quad \text{v}_j \quad \text{v}_j + 1 \quad \text{u}_j \quad \text{u}_j + 1 \quad \text{v}_j \quad \text{v}_j + 1
\]

\[
= \frac{v_j u_{j+1} - u_j v_{j+1}}{v_j(v_j + 1)} = \frac{v_j u_{j+1} - u_j v_{j+1}}{v_j(v_j + 1)}
\]

\[
= \frac{v_j u_{j+1} - u_j v_{j+1}}{v_j(v_j + 1)} = \frac{v_j (u_{j+1} - u_j)}{v_j(v_j + 1)}
\]

\[
= \frac{v_j \Delta u_j - u_j \Delta v_j}{v_j(v_j + 1)}. \quad \square
\]

(6) We now will finish the proof of (6) for \(-n\). Let \( 1_j = 1 \). Reference to (6) in the proof below refers to the first part which we have already proven.

\[
\Delta(u_j^{-n}) = \Delta \left( \frac{1}{u_j^n} \right) = \Delta \left( \frac{1}{u_j^n} \right) = \frac{u_j^n \Delta 1_j - 1_j \Delta u_j^n}{u_j^n u_{j+1}^n} = \frac{u_j^n \Delta 1_j - 1_j \Delta u_j^n}{u_j^n u_{j+1}^n}
\]

\[
= \left( \frac{u_j^n \cdot 0 - \Delta u_j^n}{u_j^n u_{j+1}^n} \right) = \frac{0 - \Delta u_j^n}{u_j^n u_{j+1}^n} = \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j)
\]

\[
= \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j) = \left( \sum_{k=1}^{n} u_j^{n-k}u_{j+1}^{k-1} \right) \cdot \Delta(u_j). \quad \square
\]
(8) \( \Delta (b^j) = b^{j+1} - b^j = b^j (b - 1) \).

(9) \( \Delta (\sin j) = \sin (j + 1) - \sin j = \sin \left( (j + \frac{1}{2}) + \frac{1}{2} \right) - \sin \left( (j + \frac{1}{2}) - \frac{1}{2} \right) \)

\[ = \sin \left( j + \frac{1}{2} \right) \cos \frac{1}{2} + \cos \left( j + \frac{1}{2} \right) \sin \frac{1}{2} - \left( \sin \left( j + \frac{1}{2} \right) \cos \frac{1}{2} - \cos \left( j + \frac{1}{2} \right) \sin \frac{1}{2} \right) \]

\[ = \sin \left( j + \frac{1}{2} \right) \cos \frac{1}{2} + \cos \left( j + \frac{1}{2} \right) \sin \frac{1}{2} - \sin \left( j + \frac{1}{2} \right) \cos \frac{1}{2} + \cos \left( j + \frac{1}{2} \right) \sin \frac{1}{2} \]

\[ = 2 \cos \left( j + \frac{1}{2} \right) \sin \frac{1}{2} = \frac{1}{2} \sin (j + \frac{1}{2}) . \]

\( \Delta (\cos j) = \cos (j + 1) - \cos j = \cos \left( (j + \frac{1}{2}) + \frac{1}{2} \right) - \cos \left( (j + \frac{1}{2}) - \frac{1}{2} \right) \)

\[ = \cos \left( j + \frac{1}{2} \right) \cos \frac{1}{2} - \sin \left( j + \frac{1}{2} \right) \sin \frac{1}{2} - \left( \cos \left( j + \frac{1}{2} \right) \cos \frac{1}{2} + \sin \left( j + \frac{1}{2} \right) \sin \frac{1}{2} \right) \]

\[ = \cos \left( j + \frac{1}{2} \right) \cos \frac{1}{2} - \sin \left( j + \frac{1}{2} \right) \sin \frac{1}{2} - \cos \left( j + \frac{1}{2} \right) \cos \frac{1}{2} - \sin \left( j + \frac{1}{2} \right) \sin \frac{1}{2} \]

\[ = -2 \sin \left( j + \frac{1}{2} \right) \sin \frac{1}{2} = -2 \sin \frac{1}{2} \sin (j + \frac{1}{2}) . \]

\( \Delta (\tan j) = \Delta \left( \frac{\sin j}{\cos j} \right) = \frac{\Delta \sin j \Delta \cos j - \sin j \Delta \cos j}{\cos j \cos (j + 1)} \)

\[ = \frac{\cos j \cdot 2 \sin \frac{1}{2} \cos \left( j + \frac{1}{2} \right) - \sin j \cdot (-2) \sin \frac{1}{2} \sin \left( j + \frac{1}{2} \right)}{\cos j \cos (j + 1)} \]

\[ = \frac{2 \sin \frac{1}{2} \cos \left( j + \frac{1}{2} \right) + 2 \sin \frac{1}{2} \sin \left( j + \frac{1}{2} \right)}{\cos j \cos (j + 1)} \]

\[ = \frac{2 \sin \frac{1}{2} \left( \cos j \cos \left( j + \frac{1}{2} \right) + \sin j \sin \left( j + \frac{1}{2} \right) \right)}{\cos j \cos (j + 1)} = \frac{2 \sin \frac{1}{2} \cos \left( j - \left( j + \frac{1}{2} \right) \right)}{\cos j \cos (j + 1)} \]

\[ = \frac{2 \sin \frac{1}{2} \cos \left( -\frac{1}{2} \right)}{\cos j \cos (j + 1)} = \frac{2 \sin \frac{1}{2} \cos \frac{1}{2}}{\cos j \cos (j + 1)} \]

\[ = \frac{\sin \left( 2 \cdot \frac{1}{2} \right)}{\cos j \cos (j + 1)} = \frac{\sin 1}{\cos j \cos (j + 1)} = \sin 1 \sec j \sec (j + 1) . \]

\( \Delta (\tan j) = \tan (j + 1) - \tan j = \frac{\tan j + \tan \frac{1}{1} - \tan j}{1 - \tan j \tan \frac{1}{1}} - \tan j \)

\[ = \frac{\tan j + \tan 1 - \tan j \left( 1 - \tan j \tan \frac{1}{1} \right)}{1 - \tan j \tan \frac{1}{1}} = \frac{\tan j + \tan 1 - \tan j + \tan^2 j \tan \frac{1}{1}}{1 - \tan j \tan \frac{1}{1}} \]

\[ = \frac{\tan 1 + \tan^2 j \tan \frac{1}{1}}{1 - \tan j \tan \frac{1}{1}} = \frac{\tan 1 (1 + \tan^2 j)}{1 - \tan j \tan \frac{1}{1}} = \frac{\tan 1 \sec^2 j}{1 - \tan j \tan j} . \]

\[ \Delta (\tan j) = \frac{\tan (j + 1) - \tan j}{1 + \tan (j + 1) \tan j} \cdot (1 + \tan (j + 1) \tan j) \]

\[ = \tan ((j + 1) - j) \cdot (1 + \tan (j + 1) \tan j) = \tan (j + 1) \cdot (1 + \tan (j + 1) \tan j) \]

\[ = \tan 1 (1 + \tan (j + 1) \tan j) = \tan 1 (1 + \tan j \tan (j + 1)) . \]
(10) $\Delta (\log_b u_j) = \log_b u_{j+1} - \log_b u_j = \log_b \left( \frac{u_{j+1}}{u_j} \right)$.

$\Delta (\ln u_j) = \Delta (\log_e u_j) = \sum \log_e \left( \frac{u_{j+1}}{u_j} \right) = \ln \left( \frac{u_{j+1}}{u_j} \right)$.

$\Delta (\log_b(\alpha + j\beta)) = \log_b \left( \frac{\alpha + (j+1)\beta}{\alpha + j\beta} \right) = \log_b \left( 1 + \frac{\beta}{\alpha + j\beta} \right)$.

$\Delta (\ln(\alpha + j\beta)) = \Delta (\log_e(\alpha + j\beta)) = \log_e \left( 1 + \frac{\beta}{\alpha + j\beta} \right)$.

$\square$

Finite Difference and the Fundamental Theorems of Calculus

Of particular interest is the correspondence between the Telescoping or Collapsing Sum of Finite Difference and the First Fundamental Theorem of Calculus:

$\sum_{j=1}^{n-1} \Delta(u_j) = u_n - u_1 = u_j^n \int_a^b D(u) \, dx = u(b) - u(a) = u^n_a$

and the correspondence between the Finite Difference of Finite Series and the Second Fundamental Theorem of Calculus:

$\Delta \left( \sum_{t=1}^{j-1} u_t \right) = u_j \quad D_x \left( \int_a^x u(t) \, dt \right) = u(x)$

Proofs

Proofs of the Fundamental Theorems of Calculus shown above can be found in any calculus text, e.g., [3].

(11) $\sum_{j=1}^{n-1} \Delta(u_j) = \sum_{j=1}^{n-1} (u_{j+1} - u_j) = \sum_{j=1}^{n-1} u_{j+1} - \sum_{j=1}^{n-1} u_j = \sum_{j=1}^{n-2} u_{j+1} + \sum_{j=1}^{n-1} u_n - \left( u_1 + \sum_{j=2}^{n-1} u_j \right)$

$= \sum_{j=1}^{n-1} u_{j+1} + \sum_{j=1}^{n-2} u_n - \sum_{j=1}^{n-1} u_j = \sum_{j=1}^{n-2} u_{j+1} + \sum_{j=1}^{n-1} u_n - \sum_{j=2}^{n-1} u_j$

$= u_n - u_1 + \sum_{j=2}^{n-1} u_j - \sum_{j=2}^{n-1} u_j = u_n - u_1 + 0 = u_n - u_1 = u_j^n_1$.

$\square$

(12) $\Delta \left( \sum_{t=1}^{j-1} u_t \right) = \sum_{t=1}^{(j+1)-1} u_t - \sum_{t=1}^{j-1} u_t = \sum_{t=1}^{j+1-1} u_t - \sum_{t=1}^{j-1} u_t$

$= \sum_{t=1}^{j} u_t - \sum_{t=1}^{j-1} u_t = \sum_{t=1}^{j} u_t - \sum_{t=1}^{j-1} u_t = u_j + 0$

$= u_j$.

$\square$
References

