Calculus Group I (1)

Let $a$ and $t$ be real numbers. Evaluate the following limit if it exists.

$$\lim_{t \to 0} \frac{(a + t)^{100} - a^{100}}{t}$$

Calculus Group I (2)

Let $f$ be a function on the real line such that $f(0) = 1$ and for all real numbers $x$ and $h$,

$$f(x + h) - f(x) = 8xh - 2h + 4h^2.$$ 

Calculate (with justification) $f(2)$, $f'(2)$, and $f''(2)$.

Calculus Group I (3)

Let $C$ be the circle in the xy-plane having center $(0, 0)$ and radius 1. A light beam shines upward and to the right from the point $(-1, 0)$ at an acute angle $A$ from the x-axis. When the beam hits the circle it is reflected (as if the circle were a mirror) to the point $(p, 0)$ on the x-axis. Find the limit of $p$ as $A$ approaches 0.

Calculus Group I (4)

Let

$$f(x) = \int_{x}^{x+\pi} |t \sin t| \ dt.$$ 

a) Evaluate $f(0)$ and $f(-\pi)$.

b) Compute $f'(x)$.

Hint: $\int_{x}^{x+\pi} |t \sin t| \ dt = \int_{x}^{0} |t \sin t| \ dt + \int_{0}^{x+\pi} |t \sin t| \ dt$.

c) Show that $f$ is increasing on the interval $[0, \infty)$ and decreasing on the interval $(-\infty, -\pi]$.

d) What is the minimum of $f$?
Calculus Group I (5)

Let $C$ be the parametrized curve in the $xy$-plane

$$C = \left\{ (x, y) \mid x = \tan(t), \ y = \tan^2(t), \text{ for } 0 \leq t \leq \frac{\pi}{4} \right\}.$$

Find the area of the region between $C$ and the $x$-axis.

Abstract Algebra Group I (6)

Let $(T, +, \cdot)$ be the ring of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, and where $+$ and $\cdot$ are the usual addition and multiplication operations on functions.

Let $S = \{ f \in T \mid f(2) = 0 \}$. Prove or disprove that $S$ is a subring of $T$.

Advanced Calculus Group I (7)

Suppose that $h$ is a real function $h : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere and satisfies the condition $h(m/2^n) = 0$ for all integers $m$ and positive integers $n$. Show that $h(x) = 0$ for all $x$ in $\mathbb{R}$.

Differential Equations Group I (8)

Solve the first order differential equation

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

subject to the initial condition $y(0) = 1$.

Discrete Mathematics Group I (9)

The graph $K_6$ consists of 6 vertices and exactly one edge between each pair of vertices. Each of these edges is to be colored one of two possible colors. In such a coloring, does there necessarily exist a triangle that is all of one color?
In other words, prove or disprove that a two-coloring of the graph \( K_6 \) contains a monochromatic subgraph isomorphic to \( K_3 \).

**Linear Algebra Group I (10)**

Suppose that \( A \) is a real \( 3 \times 3 \) matrix such that

\[
\text{the null space of } A \text{ is } \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \ a \in \mathbb{R} \right\},
\]

\[
A \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } A \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.
\]

a) What are the eigenvalues of \( A \)?

b) Find all matrices \( A \) which satisfy all three conditions above.

c) Find all real numbers \( c \) for which the limit

\[
\lim_{n \to \infty} c^n A^n
\]

exists and is a non-zero matrix.

**Abstract Algebra Group II (11)**

In the late nineteenth century the 15 puzzle (or 14-15 puzzle) became widely popular. In this game the fifteen tiles are rearranged through moves consisting of sliding (up, down, left, or right) an adjacent tile into the empty space. The goal is to arrange the tiles in numerical order as they appear on the right below. The original puzzle began with the 14 and 15 swapped, and offered a prize to anyone who could describe a series of moves that would order the pieces as those on the right. The puzzle was, of course, impossible since any series of moves that return the blank square to the lower right corner corresponds to an even permutation of the elements \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \): i.e., the permutation is an element of \( A_{15} \). A single transposition is an odd permutation and, thus, the 14 and 15 could not be swapped. You have been given a version of this puzzle. The pieces appear as shown in the square on the left. Can they
be rearranged to appear in the order shown on the right? In other words, is the appropriate permutation an element of the alternating group of 15 elements?

**Linear Algebra Group II (12)**

Let \( A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{pmatrix} \).

a) (3 pts.) Explain how you know that the eigenvalues of \( A \) are all real and that \( A \) is diagonalizable.

b) (7 pts.) Estimate (to the nearest hundredth) the dominant eigenvalue of \( A \) (i.e.: the eigenvalue with the greatest absolute value) and an associated eigenvector.

**Advanced Calculus Group II (13)**

a) Give an example of a divergent sequence of real numbers \( \{x_n\}_{n=1}^{\infty} \) which satisfies \( |x_{n+1} - x_n| < \frac{1}{n} \) for every \( n \).

b) Prove that if \( \{x_n\}_{n=1}^{\infty} \) satisfies \( |x_{n+1} - x_n| < \frac{1}{2^n} \) for every \( n \) then \( \{x_n\}_{n=1}^{\infty} \) converges to a limit.

**Advanced Calculus Group II (14)**

Let \( a_k \) be a (finite or infinite) sequence of positive numbers such that \( \sum a_k = 1 \). Explain why

\[ \sum \frac{\sqrt{a_k}}{k} < \frac{4}{3}. \]
Differential Equations Group II (15)

Consider the system of differential equations

\[
\begin{align*}
\frac{dx}{dt} &= -2x - 3y \\
\frac{dy}{dt} &= 3x - 2y
\end{align*}
\]

a) Write the general solution of this system as a linear combination

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}.
\]

b) Find the solution to the initial value problem with \( x(0) = 1 \), \( y(0) = 0 \).

c) Describe what happens to the general solution as \( t \) goes to \( \infty \).

Multivariable Calculus Group II (16)

Suppose \( \gamma \) is a curve which goes around the unit circle once in the counterclockwise direction and \( F = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \) is a vector field which is tangent to each circle of radius \( r \) about the origin but is not defined at the origin.

a) Parametrize \( \gamma \) and evaluate the integral \( \int_{\gamma} F \). (Note: this integral is also written as \( \int_{\gamma} F \cdot T \, ds \) or as \( \int_{\gamma} \left( \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy \right) \) in some texts.)

b) Use your answer to show that the vector field \( F \) is closed (that is, \( F = (M, N) \) with \( \partial N/\partial x = \partial M/\partial y \)) but is not exact (that is, there is no function \( f \) defined in the plane minus the origin where \( F = \nabla f = (\partial f/\partial x, \partial f/\partial y) \)).

c) Explain why this does not violate Green’s theorem.

Numerical Analysis Group II (17)

Tell me everything you know about Newton’s method for solving equations.
Topology Group II (18)

Let $\mathbb{R}$ be the set of real numbers and $\tau$ be the collection of subsets

$$\tau = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}.$$ 

a) Verify that $\tau$ is a topology on $\mathbb{R}$.

b) Show that the sequence $\{n\}_{n=1}^{\infty}$ converges to $z$, for any $z \in \mathbb{R}$.

c) What is the closure of the set $\{1, 2\}$?

d) What is the smallest connected set which contains $\{1, 2\}$?

Probability Group II (19)

Consider an urn which has 50 red socks and 100 blue socks in it. Suppose that 40 socks are to be drawn at random from the urn.

a) If drawing is done with replacement, then is it more likely to draw a red sock on the first draw or a red sock on the 30th draw? Explain your answer.

b) If drawing is done without replacement, then is it more likely to draw a red sock on the first draw or a red sock on the 30th draw? Explain your answer.

Set Theory/Logic Group II (20)

Let $g : A \to B$ and $f : B \to C$ be functions.

a) Prove or disprove that if $f \circ g$ is a one-to-one function, then $g$ is a one-to-one function.

b) Prove or disprove that if $g$ is a one-to-one function, then $f \circ g$ is a one-to-one function.