Calculus - Group I

Consider the circle of radius \( r = a \) centered at \((b,0)\) where \( a < b \). The equation of this circle is

\[
y^2 + (x - b)^2 = a^2.
\]

Form a torus by revolving this circle about the \( y \)-axis.

(a) Using calculus, show that the volume of the torus is \( V = 2\pi^2 a^2 b \).

(b) Give a geometric interpretation of the volume of this torus.

Note: You may receive credit for part (b) without doing part (a).

Calculus - Group I

Find the exact and simplified value of the integral

\[
\int_0^{\pi/4} \left[ \ln \left( \sin(x) + \cos(x) \right) - \ln \left( \cos(x) \right) \right] dx.
\]

Calculus - Group I

Below is a plot showing a cubic curve given by the equation \( y = x^3 \) and a straight half-line \( l \) in the first quadrant of the \( xy \)-plane. The shaded region bounded by the curve and the line has an area \( A \). Suppose that the half-line \( l \) has a hinge at the origin \((0,0)\) and is allowed to rotate counter-clockwise from the \( x \)-axis toward the \( y \)-axis while keeping the cubic curve fixed. As the line rotates around, the area \( A \) of the bounded region will grow.

Assume that \( l \) rotates counter-clockwise at a constant rate of \( 2/\pi \) revolutions per minute (rpm). How rapidly is the area \( A \) growing at the instant when the slope of \( l \) is 10? \( \text{Hint:} \) The correct final answer is a recently popular positive integer whose prime divisors are 2, 5, and 101.
Calculus - Group I

A right triangle is constructed in the first quadrant so that one vertex is at the origin, another vertex is on the positive $x$-axis, and a third vertex is on the curve $y = x^7e^{x^2}$. If the hypotenuse is the line segment from the origin to the point of the curve, find the exact value of the maximum area of such a triangle.

Calculus - Group I

Evaluate the following limit exactly.

$$\lim_{x \to 0} \frac{3x^2 - \arctan(3x^2)}{7x^6}$$

Hint: You may want to consider the Taylor series for the arctangent function.

Abstract Algebra - Group I

Consider the commutative ring $R = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$, endowed with the usual addition and multiplication operations of the real numbers.

a) Determine, with proof, whether or not $R$ is a field.

b) Determine all possible ring isomorphisms $\phi : R \to R$. Prove that your list is complete.

Differential Equations - Group I

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} - (y - x)^2; \quad y(1) = 2$$

Hint: Try the substitution $y = x + \frac{1}{v}$.

Discrete Mathematics - Group I

Any map can be colored with only 4 colors where no bordering countries will have the same color. After many failed proofs, the final proof required coming up several classes of planar graphs and a computer was used to show all cases were colorable. Before this program was created, a less strict, but easier, proof of showing every planar graph was 6 colorable. A map can be converted into a planar graph by placing a vertex in every country. Then connect pairs of vertices with an edge when two countries border each other. Every vertex in the graph can then be given a color number where no two adjacent vertices will have the same color.

A necessary, but not sufficient theorem for planar graphs is a limit on the number of edges in the graph, $|E(G)| \leq 3|V(G)| - 6$. The degree of a vertex, $d(v)$ is the number of edges attached to the vertex. The minimum degree of a graph, $\delta(G) \leq d(v_i)$ is the smallest degree of a vertex over all the vertices of the graph.
1. Prove that every planar graph must have \( \delta(G) \leq 5 \).

2. Prove all planar graphs are 6 colorable.

**Linear Algebra - Group I**

Let \( n \geq 2 \) be a given integer. Let \( A = [a_{ij}] \) be an \( n \)-by-\( n \) matrix with real entries. Set

\[
f_A(x) = x^T Ax \text{ for } x \in \mathbb{R}^n.
\]

1. Let \( B = \frac{1}{2} (A + A^T) \). Show that \( f_A(x) = f_{A^T}(x) = f_B(x) \) for every \( x \in \mathbb{R}^n \).

2. Suppose that \( A \) is skew-symmetric. Show that \( f_A(x) = 0 \) for every \( x \in \mathbb{R}^n \).

3. Let \( S \) be a 2-by-2 symmetric matrix with real entries. Show that \( f_S(x) = 0 \) for every \( x \in \mathbb{R}^2 \) if and only if \( S = 0 \).

**Multivariable Calculus - Group I**

Evaluate the following triple integral

\[
\iiint_G \sin \left( \pi \sqrt{x^2 + y^2 + z^2} \right) \, dV
\]

where \( G \) is the region in \( \mathbb{R}^3 \) enclosed by the sphere of radius \( \frac{8}{3} \) centered at the origin.

**Algebra - Group II**

Let \( A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \)

1. (2 points) Determine the eigenvalues of \( A \) and find a basis for the eigenspaces of \( A \).

2. (5 points) Determine a \( 3 \times 3 \) matrix \( B \) so that

\[
B^{-1} A B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

3. (3 points) Find (exactly)

\[
A^{50} + A^{-50}
\]
Advanced Calculus - Group II

Let $A = \{0, 1, 2, 3, 4, 5\}$. For each of the following, find (with proof) a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the given condition. (For clarification, these are three separate endeavors, no single function can fit all three parts.)

a) discontinuous at points in $A$, but continuous everywhere else

b) continuous everywhere, not differentiable at points in $A$, differentiable everywhere else

c) differentiable at points in $A$, discontinuous everywhere else

Complex Analysis - Group II

Let $f(z) = e^{iz}$.

1. (4 points) Compute the real part $u(x, y) = \text{Re}(f(z))$ and the imaginary part $v(x, y) = \text{Im}(f(z))$ of $f(z)$. Show that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations and conclude that $f(z)$ is an entire function.

2. (6 points) Let $\gamma$ be the curve with parameterization $\gamma(t) = (1 + \cos t) + i \sin t$, $0 \leq t < 2\pi$. Evaluate the following contour integrals. In each case give the real and the imaginary parts.

\[
\oint_{\gamma} \frac{e^{iz}}{z-1} \, dz
\]

Differential Equations - Group II

Show that $y_1(x) = x$ satisfies the following differential equation:

\[x^2 y'' - 3xy' + 3y = 0.\]

Find a nontrivial solution of the form $y = uy_1$, with $u$ not a constant.

Geometry - Group II

A line $\lambda$ is tangent to the incircle of an equilateral triangle $ABC$. Also, the line $\lambda$ is perpendicular to the side $AB$, intersects the side $BC$ (between the points $B$ and $C$) at the point $D$, and intersects the side $AB$ at the point $E$. If the segment $DE$ has length 1 unit, what is the exact area of triangle $ABC$? Simplify your answer.
Linear Algebra - Group II

Suppose $A$ is a $3 \times 3$ matrix with real entries. It is known that 3 and $3 + 2i$ are two of its eigenvalues. Find the determinant and the trace of the matrix $A$. Provide the exact and simplified answers.

Number Theory - Group II

A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal parts, 3 coins remained. In the ensuing brawl to see who got the extra coins, 1 pirate was killed. The wealth was redistributed, but this time an equal division left 10 coins. In the ensuing brawl, 1 more pirate was killed. Now the coins could be evenly distributed among the surviving 15 pirates. What is the least number of coins that could have been stolen?

Hint: The Chinese Remainder Theorem may be useful.

Point-Set Topology - Group II

For the set $\mathbb{R}$ of all real numbers, define topologies $T_1$, $T_2$, $T_3$, and $T_4$ as follows.

1. $T_1$ consists of all sets which are unions of half-open intervals of the form $(a,b]$.
2. $T_2$ consists of all sets which are unions of closed rays of the form $(-\infty, b]$.
3. $T_3$ consists of the empty set, the set of all rational numbers, the set of all irrational numbers, and $\mathbb{R}$.
4. $T_4$ consists of the empty set and all subsets of $\mathbb{R}$ which have a finite complement.

When $\mathbb{R}$ is given the usual topology, the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ converges to 0 and only to 0. To what points, if any, does this sequence converge when $\mathbb{R}$ is given the four topologies above? Provide a proof for one of the four results.

Probability/Statistics - Group II

An industrial robot has been designed to cut a one-meter rod into three pieces according to the following algorithm: First the robot chooses a cutting point $X$ somewhere along the length of the rod, completely at random and with a continuous uniform distribution over the entire rod. A cut is made at the point $X$, dividing the rod into the two pieces $[0,X]$ and $[X,1]$. Then the robot takes the left-most piece $[0,X]$ and chooses a second cutting point $Y$ at random, uniformly distributed from 0 to $X$. A cut is made at $Y$, resulting in the three pieces $[0,Y]$, $[Y,X]$, and $[X,1]$. The robot then places the $[0,Y]$ piece into a bin labeled “Left”, the $[Y,X]$ piece into a bin labeled “Middle”, and the $[X,1]$ piece into a bin labeled “Right”. Note that the second cut-point $Y$ is always to the left of the first cut-point $X$, and that the lengths of the three pieces are random and sum to 1.

Suppose that the robot repeats this process on a million rods, with the random choices for $X$ and $Y$ on each rod being independent of the cut-locations on the other rods. And the end of this process, there are a million pieces sitting in the “Left” bin. A probabilist colleague of yours has told you that if you calculate the arithmetic average of the lengths of the million pieces in the “Left” bin, the average length will be very close to 0.25 meters, but that half of the Left pieces will be shorter than 0.1867 meters. Is she correct?
In a Futurama episode, the professor invented a mind swapping machine. After testing the machine, they found out they were not able to switch back to their bodies. So they swapped with other people and decided to ignore the problem for now. After a while, nobody was in their right body. It turned out that after swapping minds, the machine would not allow the same pair of minds to swap again. Is there a way to get everyone back into their own bodies given the limitations of the machine? You should assume everyone from A through G has already swapped minds, so you will need to bring in additional (un)willing participant(s) who never used the machine before to bypass the security. Show the list of mind swaps necessary to get everyones mind back into their own bodies, including the additional (un)willing participant(s).

<table>
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<tr>
<th>Body</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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<th>G</th>
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</thead>
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<tr>
<td>Mind</td>
<td>G</td>
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<td>C</td>
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