NOTES FOR A LECTURE TO THE M.A.A. & IOWA ACADEMY

Infinitesimal Analysis - Then and Now. K.D. Stroyan, Math. Sci. Div., Univ. of Iowa, Iowa City, Iowa 52242.

Abraham Robinson's modern theory of infinitesimals ("nonstandard analysis") gives us a new perspective from which to view the history of calculus. We can interpret old proofs of Euler, Cauchy or Gauss and see what changes are needed to make them rigorous in the current sense. We can make interesting comparisons with epsilon-delta modernizations of those same proofs. Frequently the two modernizations are inequivalent! (Both are modifications and incomparable to the originals in the strictest historical sense.) Occasionally this process has useful applications in the classroom. Some of the neoclassical proofs bring helpful geometric reasoning into basic facts of calculus. In those cases instructors have a choice of either saying the pictures are "only heuristic" or of introducing the infinitesimal numbers by some simple means. I will give Keisler's axioms for hyperreal numbers and discuss several of these new old proofs indicating why they are rigorous (or rigorizable).

LEIBNIZ' PRINCIPLE:

The ideal numbers are governed by the same laws as the ordinary numbers. (Cf. Robinson [1966, p. 261-263].)

ROBINSON-KEISLER FORMULATION OF THE PRINCIPLE:

(X-rated version, for mathematical adults—it needn't be so "heavy," cf. Keisler [1976].)

Algebraic Axiom (Group):

The hyperreal numbers $^*\mathbb{R}$ are a proper ordered field extension of Dedekind's numbers $\mathbb{R}$. (Proper extension implies the existence of positive and negative infinite and infinitesimal numbers.)

Definition: Let ${}^{\sigma}\mathbb{R}$ denote the embedded (standard) numbers, ${}^{\sigma}\mathbb{R} \subset {}^{*}\mathbb{R}$.

(FIN & INF)

A number x in ${}^{*}\mathbb{R}$ is finite if $|x| < r$ for some r in ${}^{\sigma}\mathbb{R}$, otherwise it is infinite, specifically, Ω is infinite if $|\Omega| > r$ for all $r \in {}^{\sigma}\mathbb{R}$. The ring of finite numbers is denoted $\mathcal{O}$ (big oh—like Landau's).

(INFSML)

A number δ in ${}^{*}\mathbb{R}$ is infinitesimal if $|\delta| < \epsilon$ for every positive ϵ in ${}^{\sigma}\mathbb{R}$. Archimedes axiom says that zero is the only infinitesimal in ${}^{\sigma}\mathbb{R}$. We write $x \approx y$ if $x - y$ is infinitesimal.

ALGEBRAIC CONSEQUENCES:

(X-rated): The finite Robinson numbers are an ordered ring containing the infinitesimals as a maximal order ideal. The quotient $\mathcal{O}/\approx$ is isomorphic to $\mathbb{R}$ and each cofactor contains a unique embedded Dedekind number $r \in {}^{\sigma}\mathbb{R}$ called the standard part of a finite x, $st(x) = {}^{\circ}x = \hat{x} = r$.

(PG-rated): Finite times infinitesimal is infinitesimal, reciprocal of infinite is infinitesimal, etc., but you still cannot divide by zero and infinite times infinitesimal is indeterminant BUT SOMETIMES DETERMINABLE:

$$\frac{(x+\delta)^3 - x^3}{\delta} = \frac{x^3 + 3x^2\delta + 3x\delta^2 + \delta^3 - x^3}{\delta} = 3x^2 + 3x\delta + \delta^2$$

$$\approx 3x^2 \quad \text{for} \quad x \text{ finite and } \delta \neq 0 \text{ infinitesimal.}$$

Function Axiom (Scheme):

Each real-valued function f of n real variables has a natural extension $^*f$. The natural extensions of functions have the property that systems of equations and inequalities are equivalent if and only if their natural extensions are.

EXAMPLE: The addition formula for sine always holds, so it is equivalent to $[\alpha = \alpha \ \& \ \beta = \beta]$. In $^*\mathbb{R}$ this says that for all α, β in $^*\mathbb{R}$,

$$^*\sin(\alpha+\beta) = {}^*\sin(\alpha)\,{}^*\cos(\beta) + {}^*\sin(\beta)\,{}^*\cos(\alpha).$$

It is customary to drop the $*$'s on functions and numbers, since $b = f(a)$; a, b in $\mathbb{R}$ implies $^*b = {}^*f(^*a)$; $^*a, {}^*b$ in $^\sigma\mathbb{R} \subseteq {}^*\mathbb{R}$.

ALGEBRAIC AND FUNCTIONAL INFINITESIMALS ($\approx$ Leibnizian and Eulerian degrees):

The following list of infinitesimals have infinite ratios:

$$\delta^\Omega < \delta^\omega < \delta^n < \delta^3 < \delta^2 < \delta < \delta^{1/2} < \delta^{1/3} < \delta^{1/n} < \delta^{1/\omega} < \delta^{1/\Omega}$$

$$e^{-1/\delta} < \delta^n < \delta^{1/n} < \log(1/\delta)$$

where δ is positive infinitesimal, $f(x,m) = x^m$, ω is an infinite integer, say $\omega = {}^*[\lambda]$ and $\Omega = \omega^\omega$.

(Cf. Bos [1974, pp. 84-85], exercise: Robinsonize Euler's proof.)

EXERCISE: For which x and δ in ${}^*\mathbb{R}$ is

$$\frac{\sqrt{x+\delta}-\sqrt{x}}{\delta} = \frac{1}{\sqrt{x+\delta}+\sqrt{x}} \approx \frac{1}{2\sqrt{x}}\ ?$$

(In particular, where is $\sqrt{x}$ defined?)

THE INCREMENT PRINCIPLE:

(Cf. Barwise [1977, p. 208] and Nelson [1977, "derivable"] or Stroyan and Luxemburg [1976].)

Algebraic Definition.

We derive a function $f(x)$ by finding a function $f'(x)$ such that

$$f(x+\delta) - f(x) = f'(x)\cdot\delta + \delta\cdot o, \quad \text{for some } o \approx 0,$$

whenever $\delta \approx 0$ and x is finitely inside the domain of $f(x)$.

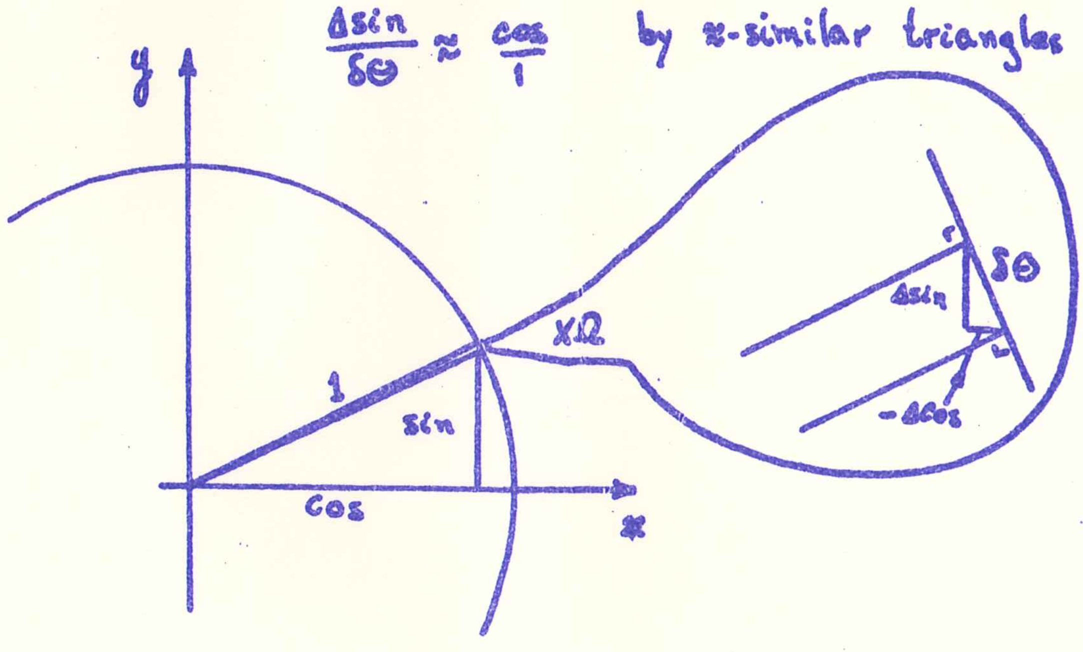
NOTE: x is finite but not necessarily Dedekind. As stated, derivable is equivalent (for standard functions) to: "continuously differentiable and defined on an open set."

Geometric Formulation.

(Cf. Hospital [1696 and 2nd ed. 1715, Demande on supposition II] and Robinson [1966, pp. 264-265].)

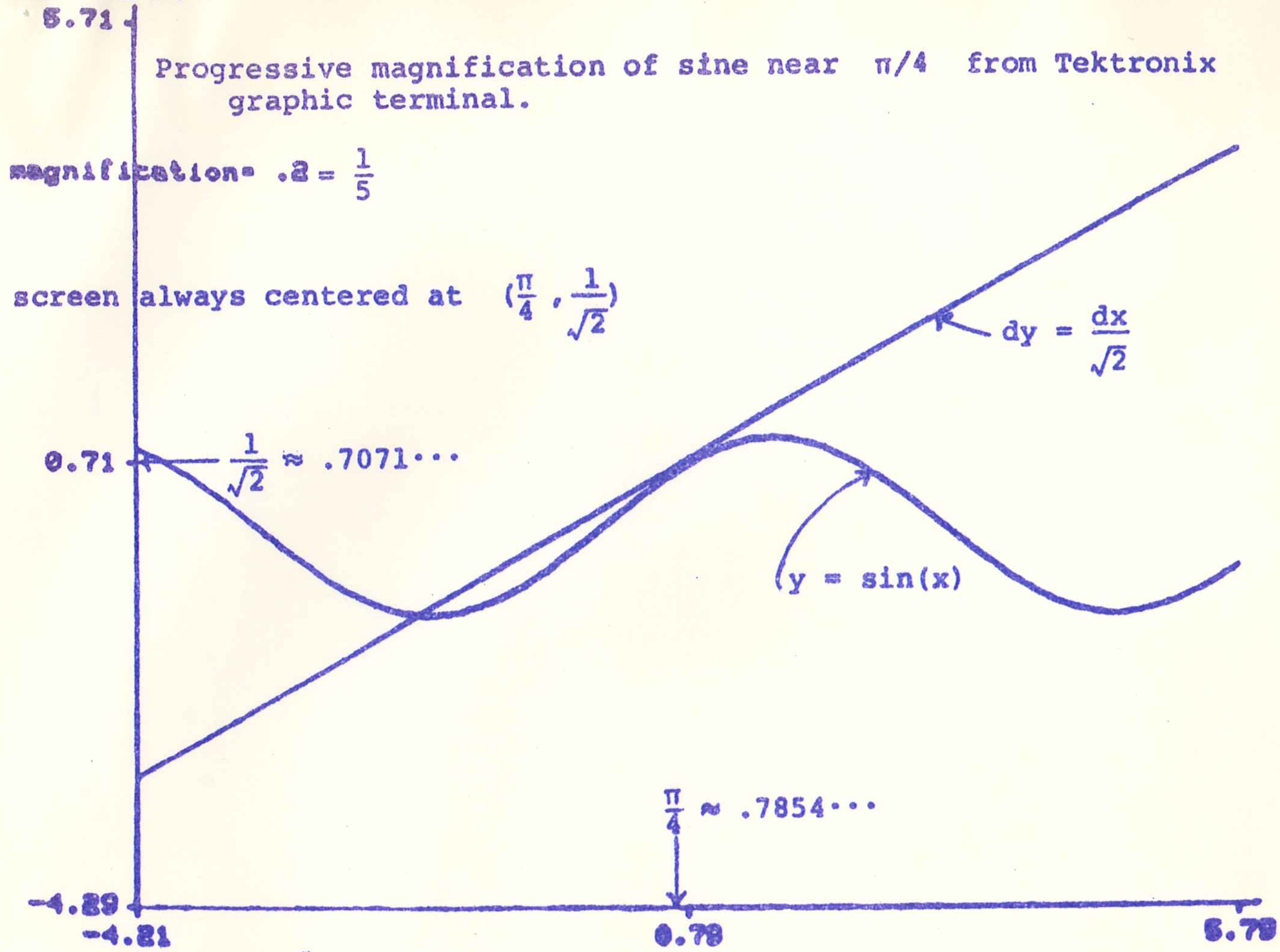
<u>Under infinite magnification a derivable curve is indistinguishable (to a finite observer) from a straight line.</u> (Cf. M.A.A. Monthly, vol. 84, nr. 6, June 1977.) Magnification is the mapping $(x,y) \longrightarrow (\frac{x-a}{\delta}, \frac{y-b}{\delta})$. "Finite observers" only see finite quantities mod $\approx$.

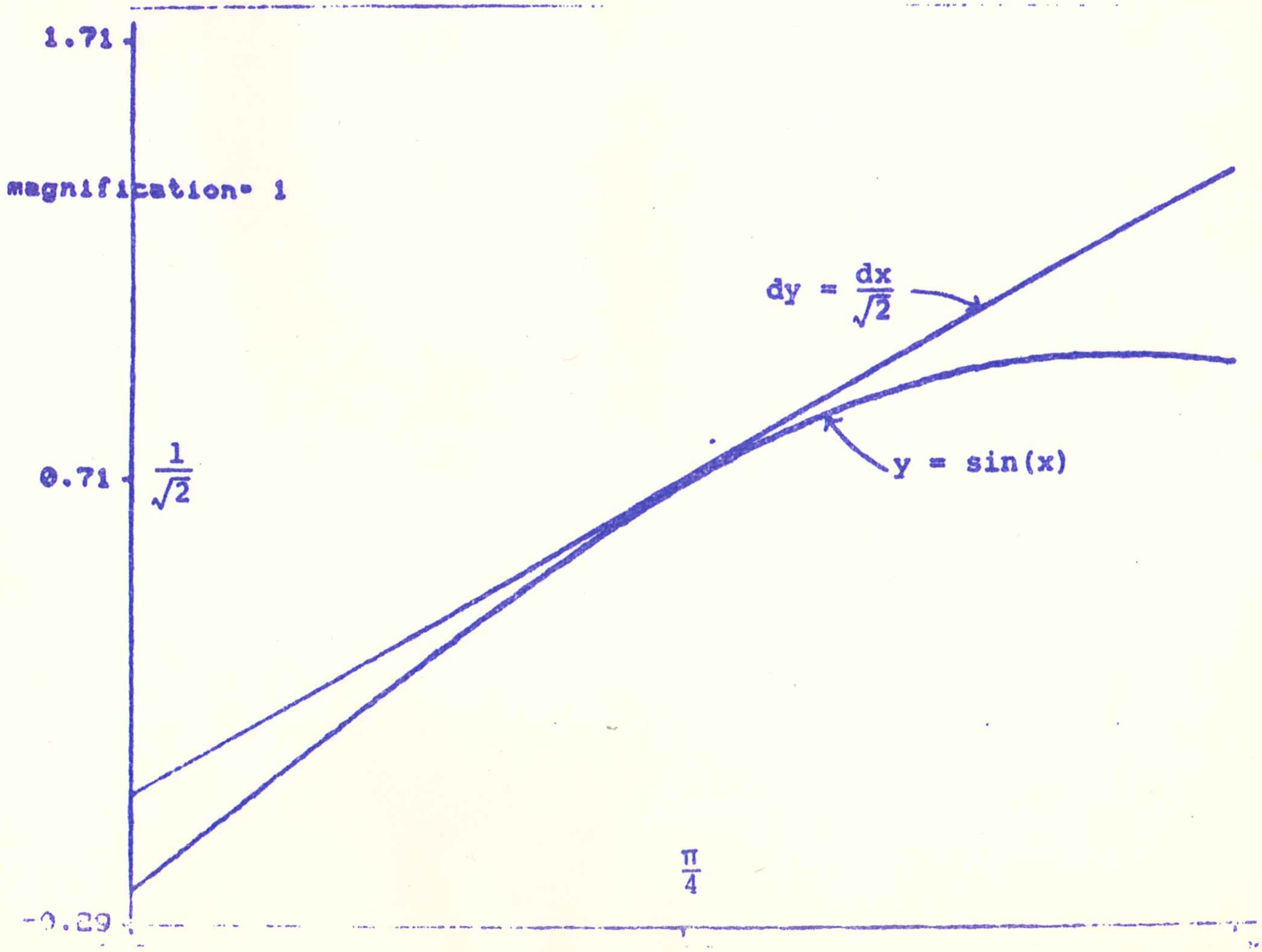
<u>DERIVING SINE AND COSINE:</u> (Note: x^2 and $x^{1/2}$ are done above. A calculation with these shows the right angles claimed.)

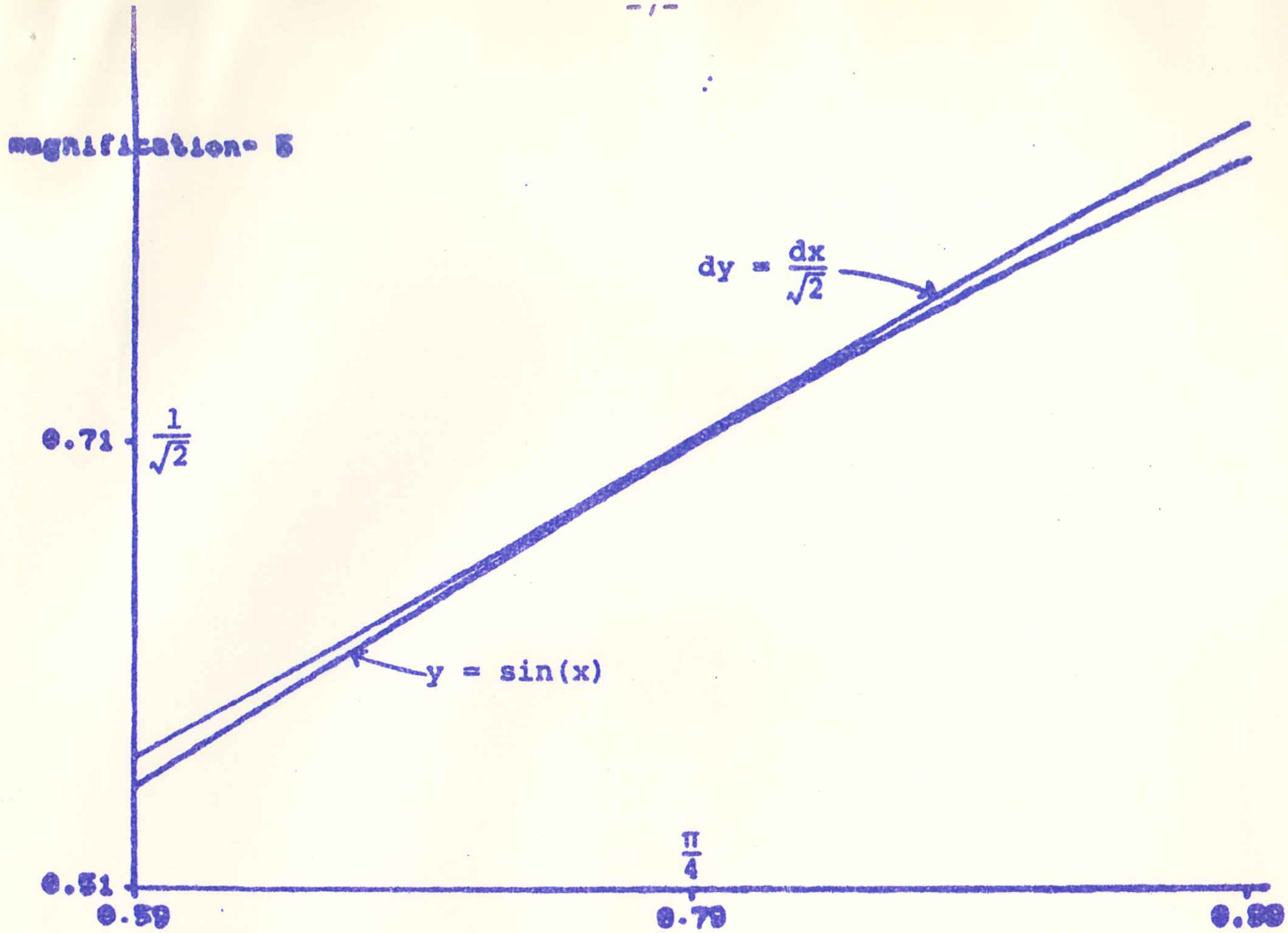


The following pages show a computer infinite microscope.

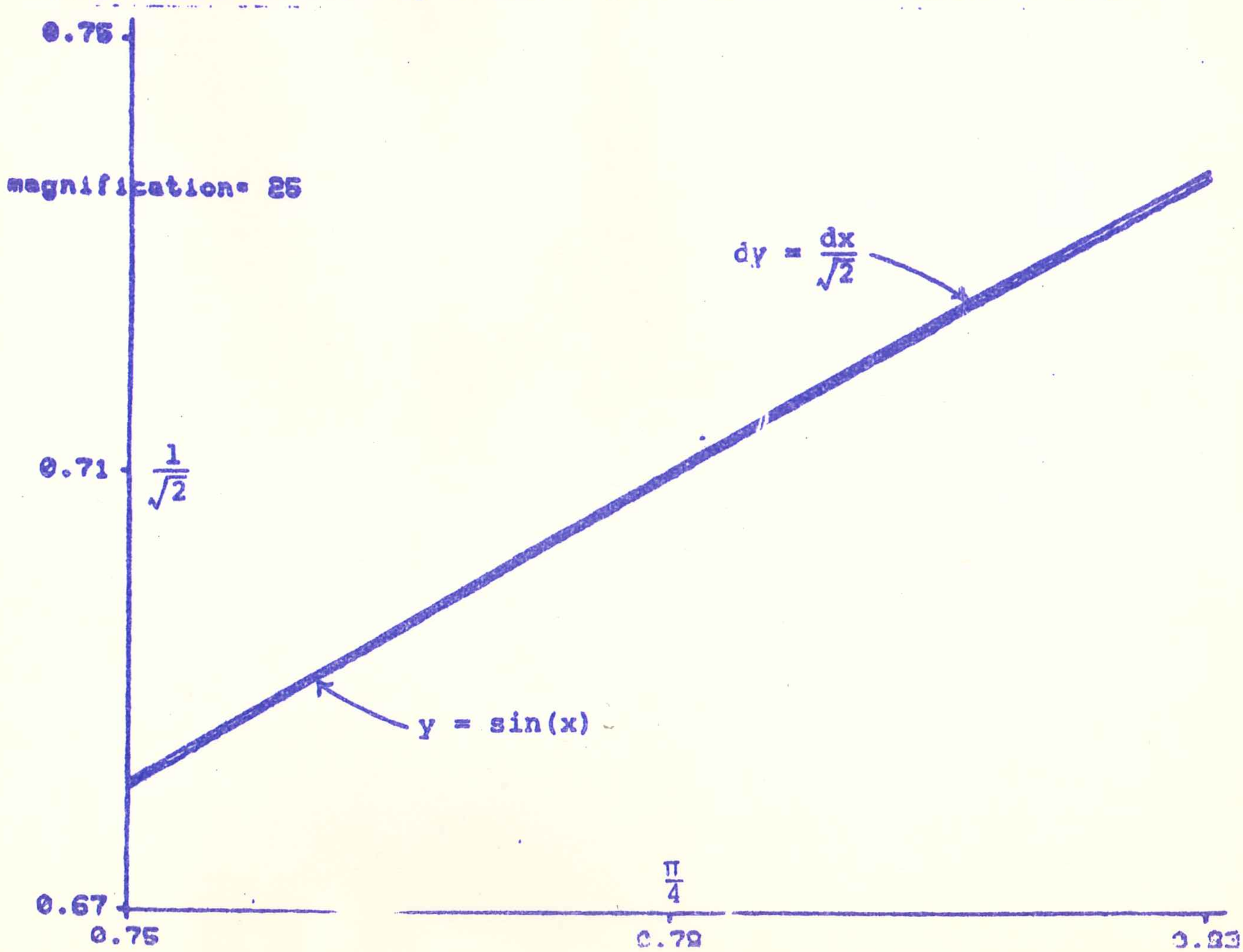
Progressive magnification of sine near π/4 from Tektronix graphic terminal.







NOTE: Line shifted up about a "pen" width — unexplained bug...



NELSON'S SUOUNITNOC FUNCTIONS:

This is example 3 from page 1177 of the article: 'INTERNAL SET THEORY', by Edward Nelson, in the Bulletin of the A.M.S., vol. 83, Nov. 1977.

'EXAMPLE 3. Which is more intuitive, the conventional definition of continuity or the nonstandard definition? Here is an unfamiliar notion with the same level of difficulty as the notion of continuity. We present it in two equivalent forms:

(i) A function $f:R \to R$ is suounitnoc at the point x in case for all ε there is a δ such that for all y, if $|y-x| \leq \varepsilon$ then $|f(y)-f(x)| \leq \delta$.

(ii) A standard function $f:R \to R$ is suounitnoc at a standard x in R if for all y in *R with (y-x) finite, then (*f(y)-*f(*x)) is also finite.

Using either definition, show that if a function is suounitnoc at one point of R then it is suounitnoc at all points of R.'

NOTE. It is conventional not to write the * on a fct. f since it is an extension of the function, b=f(a) implies *b=*f(*a)...

PROOF IN *R:

Suppose f satisfies (ii) at x in R and z is another point in R. Clearly, y-z is finite if and only if y is finite if and only if y-x is finite, since the ring 0 contains R. Thus (ii) simply says "y finite implies f(y) finite", a condition independent of x.

EXAMPLE:

$$f(x) = \begin{cases} 0, & x = 0 \\ 1/x, & x \neq 0 \end{cases}$$ is not suounitnoc

STANDARD REFERENCES:

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Martin Davis, APPLIED NONSTANDARD ANALYSIS, Wiley-Interscience Series on Pure and Applied Math., 1977.

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