Continuity and Differentiability of a Two-Variable Rational Function at the Origin

by

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Ground Rules: Without saying so explicitly, we define every function f in this paper to equal 0 at the origin, i.e., $f(0, 0) = 0$. All limits $\longrightarrow$ of functions are taken as $(X, Y) \longrightarrow (0, 0)$.

Part I. Continuity.

1. Standard Example in Multi-Variable Calculus. $\dfrac{XY}{X^2 + Y^2}$ is not continuous at $(0, 0)$. Path $Y = X$.

2. The function $\dfrac{X^2 Y}{X^4 + Y^2} \longrightarrow 0$ along all lines $Y = MX$ (in fact, it $\longrightarrow 0$ along all curves $Y = MX^r$, for $0 < r < 2$.) But the function equals $1/2$ on the parabola $Y = X^2$ (except, of course, at $(0, 0)$.)

3. Problem. For what values of $P, Q > 0$, and $M, N \geqslant 0$ is the function

$$\text{(A)} \qquad f(X, Y) = \frac{X^M Y^N}{|X|^P + |Y|^Q} \qquad \text{continuous at } (0, 0)\ ?$$

Remarks: (i). We use absolute values in the denominator to maximize the possibility of the limit of the function existing as $(X, Y) \longrightarrow (0,0)$. The denominator is positive everywhere except at $(0, 0)$.

(ii). If M or N is negative, no limit exists, as $(X, Y) \longrightarrow (0, 0)$. For example, if $M > 0$ and $N = -R < 0$, then $f \longrightarrow$ infinity along the path $Y^R = X^M$.

Theorem 1. The function f defined in (A) ....

(a.) is continuous at $(0, 0)$ iff $\dfrac{M}{P} + \dfrac{N}{Q} > 1$,

(b.) is bounded $|f| \leqslant 1$ in the plane if $\dfrac{M}{P} + \dfrac{N}{Q} = 1$,

(c.) is unbounded near $(0, 0)$ if $\dfrac{M}{P} + \dfrac{N}{Q} < 1$.

Proof: (a.) Let $\varepsilon = \dfrac{M}{P} + \dfrac{N}{Q} - 1 > 0$.

$$\left|\frac{X^M Y^N}{|X|^P + |Y|^Q}\right| = \left|\frac{X^M Y^N (\qquad)^{\varepsilon}}{(|X|^P + |Y|^Q)^{M/P}(\qquad)^{N/Q}}\right| \leq$$

$$\left|\frac{X^m}{|X|^m} * \frac{Y^N}{|Y|^N}\right| \left(|X|^P + |Y|^Q\right)^{\varepsilon} \longrightarrow 0.$$

I.3.......(b.) In the proof of part (a.), the quantity $\mathcal{E} = 0$. Clearly, $|f|$ is bounded above by 1. To show that f is not continuous at (0, 0), use the path $X^P = Y^Q$.

(c.) Use the path $X^P = Y^Q$ to show that f is unbounded near (0, 0).

Examples for Theorem 1. (a.) $\dfrac{X^2 Y}{X^2 + Y^2}$ (b.) $\dfrac{X\,Y}{X^2 + Y^2}$ (c.) $\dfrac{X\,Y^{2/3}}{X^2 + X^2}$.

4. Consider the function

(B) $$f(X, Y) = \frac{X^M Y^N}{\left(|X|^P + |Y|^Q\right)^R}$$

for P, Q, R $>$ 0, M, N $\geqslant$ 0. Consider $\left[f(X, Y)\right]^{1/R}$ and apply Theorem 1, obtaining

Theorem 2. f defined in formula (B) is continuous at (0, 0) iff

$$\frac{M}{P\,R} + \frac{N}{Q\,R} > 1.$$

Examples for Theorem 2. (R = .5) (a.) $\dfrac{X\,Y}{\sqrt{X^2 + Y^2}}$, (b.) $\dfrac{X\,Y}{\sqrt{X^4 + Y^4}}$,

(c.) $\dfrac{X\,Y}{\sqrt{X^4 + Y^6}}$.

Part II. Differentiability. Look up the definition of " f(X, Y) is differentiable at the point (a, b) " in a calculus book. Show that this definition is equivalent to

Definition 1. The function f, as defined by formula (A), is differentiable at (0, 0) if

$$\lim_{(x,y)\to(0,0)} \frac{X^M Y^N}{\left(|X|^P + |Y|^Q\right)} \cdot \frac{1}{\sqrt{X^2 + Y^2}} = 0.$$

It is well known that, if a function g is differentiable at (a, b), then g is continuous there.

Theorem 3. Define f as in formula (A), and assume, without loss of generality, that Q $\geqslant$ P. (Otherwise, just interchange the roles of X and Y.)

(I) If N $\geqslant$ 1, then f is differentiable at (0, 0) iff

$$\frac{M}{P} + \frac{N - 1}{Q} > 1.$$

(II) If $0 \leqslant N < 1$, then f is differentiable at (0, 0) iff

$$\frac{M + N - 1}{P} > 1.$$

Proof (partial): To prove that the function is not differentiable when the conditions in part (I) or in part (II) are not met, use the path $X^P = Y^Q$ for part (I), and use the path Y = X for part (II).

Definition 2. A function g(X, Y) is continuously differentiable at the point (a, b) if the partial derivatives g_x and g_y exist throughout some neighborhood of (a, b) and are continuous at the point (a, b).

II.....Theorem 4. Define f as in formula (A).

(I) The partial derivative f_X is continuous at $(0, 0)$ iff

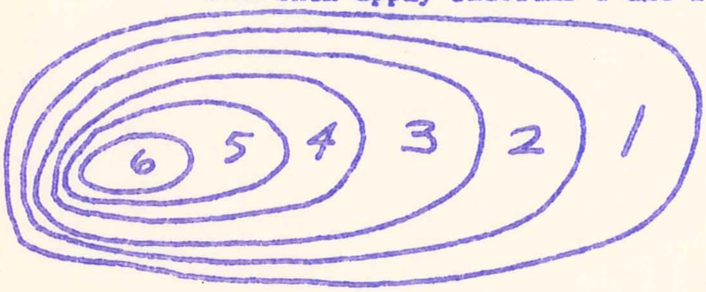
$\frac{M-1}{P} + \frac{N}{Q} > 1$ and either of the two conditions (a) or (b) holds:
(a) $M \geq 1$, (b) $M = 0$ and $P \geq 1$.

(II) The partial derivative f_Y is continuous at $(0, 0)$ iff

$\frac{M}{P} + \frac{N-1}{Q} > 1$ and either of the two conditions (c) or (d) holds:
(c) $N \geq 1$, (b) $N = 0$ and $Q \geq 1$.

Proof: The proofs of the two parts of this theorem are similar. To prove part I, just find, by differentiating the quotient, the partial derivative f_X at an arbitrary point $(X, Y) \neq (0, 0)$. Under the conditions stated, $f_X(0, 0)$ will equal 0, as a special calculation will reveal.

Then apply Theorems 1 and 2 to show that $\lim_{(X,Y)\to(0,0)} f_X(X, Y) = f_X(0, 0) = 0$.



Examples. We give six examples of functions of type (A) which satisfy, or fail to satisfy, various continuity and differentiability criteria. The number of each example places the function in the appropriate set in the above diagram.

1. $X^{2/3} Y^{2/3} / (X^2 + Y^2)$. Unbounded near $(0,0)$. Path $Y = X$.

2. $XY / (X^2 + Y^2)$. Bounded in the XY - plane, but not continuous at $(0, 0)$. Path $Y = X$.

3. $X^{4/3} Y / (X^2 + Y^2)$. Continuous but not differentiable at $(0, 0)$. Path $Y = X$.

4. $X^{2/3} Y^{2/3} / (X^{2/9} + Y^{2/9})$. Differentiable at $(0, 0)$, but neither f_X nor f_Y is continuous at $(0, 0)$. Paths $Y^{2/3} = X^{1/9}$ and $X^{2/3} = Y^{1/9}$, respectively.

5. $X^{2/3} Y / (X^{2/9} + Y^{2/9})$. Differentiable at $(0, 0)$, f_Y is continuous there, but f_X isn't. Path $Y = X^{1/9}$.

6. $X^2 Y^2 / (X^2 + Y^2)$. Continuously differentiable at $(0, 0)$. This means that both partials f_X and f_Y are continuous there.

In the diagram, the union of all six sets is the set of all functions of type (A); the union of sets 3, 4, 5, and 6 is the set of all type (A) functions which are continuous at $(0, 0)$; the union of sets 4, 5, and 6 is the set of all type (A) functions which are differentiable at $(0, 0)$, etc.