2/27/74

Dear Prof. Gillam,

I thought you would like to see the article I wrote about mathematical induction and the Twelve Days of Christmas. Robert A. Newell, Riverside High School, Milan, Washington, has written a similar article, entitled "The Twelve Days of Christmas." This article appears in The Mathematics Teacher journal for December, 1973, pages 707-708.

Perhaps I should have waited until next Christmas before sending this to you, but I decided to send it now while postage is still 8¢.

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$$(n+1)\cdot\frac{n\cdot(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}$$

$$\frac{n(n+1)}{6}(3n+3-2n-1)$$

$$\frac{n(n+1)}{6}(n+2)$$

# Mathematical Induction, Two Turtle Doves, and A Partridge in a Pear Tree

by

George Bridgman

"On the first day of Christmas, my true love sent to me
A partridge in a pear tree.

On the second day of Christmas, my true love sent to me
Two turtle doves,
And a partridge in a pear tree.

On the third day of Christmas, my true love sent to me
Three French hens,
Two turtle doves,
And a partridge in a pear tree."

....and so it goes, until....

"On the twelfth day of Christmas, my true love sent to me...."

....a grand total of 78 gifts, consisting of....

| | | | |
|---|---|---|---|
| 12 | drummers drumming, | 11 | pipers piping, |
| 10 | lords a-leaping, | 9 | ladies dancing, |
| 8 | maids a-milking, | 7 | swans a-swimming, |
| 6 | geese a-laying, | 5 | gold rings, |
| 4 | calling birds, | 3 | French hens, |
| 2 | turtle doves, and | a | partridge in a pear tree. |

Naturally the question arises -- how many gifts did the composer accumulate during the entire twelve days of Christmas?! We observe that on the n-th day of Christmas, where n may be any integer from 2 to 12, he received n identical gifts of a new type, plus all the gifts he had already been given on the previous day. Thus on the n-th day alone he collected $S_n$ gifts, where:

(1) $$S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} .$$

We may prove formula (1) by using a technique known as "mathematical induction."

At the end of the n-th day of Christmas, the composer had amassed a total of $C_n$ gifts, where:

(2) $$C_n = S_1 + S_2 + S_3 + \ldots + S_n = 1 + 3 + 6 + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6} .$$

During the twelve-day period he therefore accumulated

$$C_{12} = \frac{(12)(13)(14)}{6} = 364$$ gifts, comprising

(3)

| | | | | |
|---|---|---|---|---|
| 12 | drummers drumming | and | 12 | partridges in pear trees, |
| 22 | pipers piping | and | 22 | turtle doves, |
| 30 | lords a-leaping | and | 30 | French hens, |
| 36 | ladies dancing | and | 36 | calling birds, |
| 40 | maids a-milking | and | 40 | gold rings, |
| 42 | swans a-swimming | and | 42 | geese a-laying. |

An elementary mathematical induction argument establishes equation (2).*

We may prove that $C_n = \frac{n(n+1)(n+2)}{6}$ in a "constructive" fashion by using equation (1) together with another

* Non-mathematicians may omit reading the rest of this paper!

well-known mathematical induction formula:

(4) $$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} .$$

We have: $$C_n = S_1 + S_2 + S_3 + \dots + S_n = \sum_{k=1}^{n} \frac{k(k+1)}{2} =$$

$$= \sum_{k=1}^{n} \frac{k^2}{2} + \sum_{k=1}^{n} \frac{k}{2} =$$

$$= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} = \frac{n(n+1)(n+2)}{6} .$$

We may derive another formula for $C_n$ by considering the total number of gifts of each kind, as listed in (3). The total number of gifts received during the twelve days of Christmas is

$$1 \cdot 12 + 2 \cdot 11 + 3 \cdot 10 + \dots + 10 \cdot 3 + 11 \cdot 2 + 12 \cdot 1 = \sum_{k=1}^{12} k(13-k) .$$

For n days of Christmas (n any positive integer), the total number of gifts accumulated up to that point is

(5) $$C_n = \sum_{k=1}^{n} k(n+1-k) .$$

We may employ mathematical induction to prove the equality

(6) $$\sum_{k=1}^{n} k(n+1-k) = \frac{n(n+1)(n+2)}{6} .$$

The two formulas for $C_n$, (2) and (5), are equivalent.

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