Isoderivative Curves

by

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Consider the one-parameter family of parabolas

(1) $$y = C x^2 ,$$

where C is any real number. (If $C = 0$, the "parabola" is a horizontal line.) We differentiate each parabola, $y' = 2Cx$. On each parabola ($C \neq 0$) there is a point where the derivative $y' = 1$. To find this point, we write

(3) $$1 = 2Cx .$$

Then $x = 1/2C$. The point (x, y) where the derivative equals 1 on the curve having parameter value C has coordinates

(4) $$x = 1/2C , \qquad y = C x^2 = 1/4C .$$

If we allow C to take on all nonzero real values, then equations (4) are the parametric equations for the set of all points where any curve $y = C x^2$ has slope 1. We eliminate C between equations (4), or between (1) and (3), and find this set of points to be the line $y = x/2$, minus the origin.

We call this set of points an "isoderivative curve." (See figure on page 3.)

Just as isobars and isotherms on a weather map join points of equal pressure and equal temperature, respectively, so an isoderivative curve joins points having "equal derivative," in a one-parameter family of functions.

For family (1), the isoderivative curve connecting all points where the derivative of any function equals V is the line $y = Vx/2$, minus the origin. (If $V = 0$, the origin is part of the isoderivative curve.)

Definition: An isoderivative curve is the set of all points $P:(x, y)$ where any function in a one-parameter family has its derivative equal to a fixed value, V.

We will refer to an isoderivative curve as an "I-curve" from now on.

Interestingly, the I-curve (with slope value V) for the cubic family

(5) $$y = C x^3$$

is also a line, $y = Vx/3$, minus the origin. (The origin is included if $V = 0$.)

We verify this by eliminating C between (5) and $y' = V = 3C\,x^2$. Moreover, all "fixed power" families $y = C\,x^r$, $r \neq 0$ fixed, have straight lines for I-curves. These are $y = Vx/r$, minus the origin. (Origin included if $V = 0$.)

At the end of this paper we list a variety of function families and their I-curves.

Isoderivative Curves and the Direction Field. In differential equations we encounter a "direction field" or "slope field" or "field of tangents." A first-order differential equation

(6) $$y' = \phi(x, y)$$

establishes at each point (x, y) a "slope" having value $\phi(x, y)$. (Imagine a tiny line passing through (x, y) with slope $\phi(x, y)$.) A solution $f(x, y, C) = 0$ to (6) is any curve which joins together some collection of points (x, y) so that the curve is tangent to the line at (x, y). In contrast, an I-curve connects all points (x, y) whose lines have a common slope, V, and are therefore parallel. Generally the I-curve is not tangent to these lines. Family (1) is the solution of $y' = 2y/x$.

To obtain the differential equation (6) corresponding to a given I-curve, replace V by y' in the curve equation, then solve for y' if possible. Conversely, to get the I-curves for the family of solutions to (6), replace y' by V in (6).

Inverse Problem. We consider the following inverse problem. What family

(7) $y = C\,f(x)$ of differentiable curves has

(8) $y = V\,g(x)$ as the equation of its I-curves?

Differentiating (7) and replacing y' by V produces $V = C\,f'(x)$, or

(9) $$V = y\,f'(x)/f(x) \,,$$

using (7). We have, from (8),

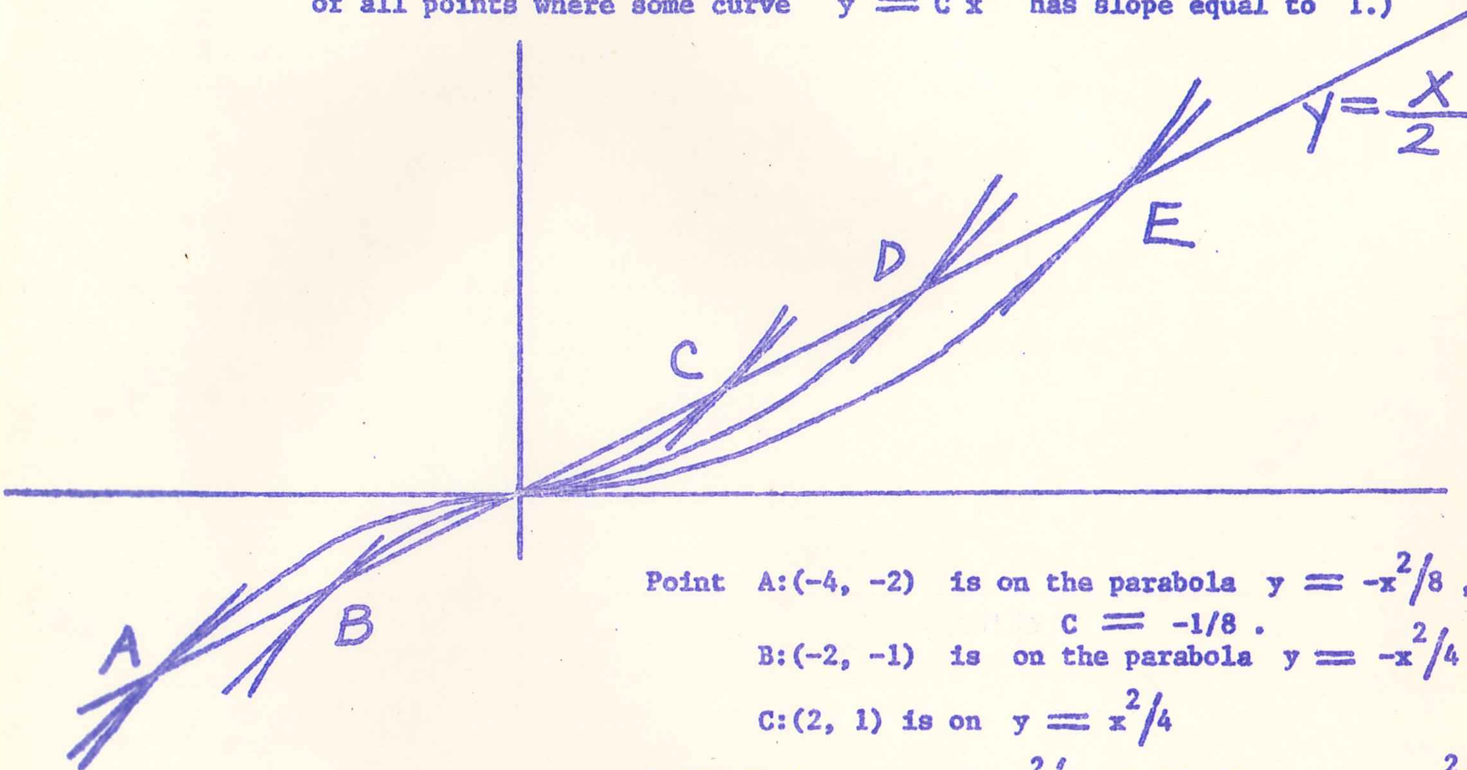
(10) $V = y/g(x)$. Hence $g\,f' = f$, by (9) and (10).

Thus we seek a function $z = f(x)$ satisfying the first-order linear differential equation $z' - z/g(x) = 0$. Family (7) is $y = C e^{\int \frac{1}{g(x)} dx}$.

We list some examples of the inverse problem at the end of the table.

The Family of Parabolas $y = C x^2$ and the Isoderivative Curve $y = x/2$ for Derivative Value $V = 1$

(The line $y = x/2$ (excluding the origin) consists of all points where some curve $y = C x^2$ has slope equal to 1.)



Point A: $(-4, -2)$ is on the parabola $y = -x^2/8$, $C = -1/8$.
B: $(-2, -1)$ is on the parabola $y = -x^2/4$.
C: $(2, 1)$ is on $y = x^2/4$
D: $(4, 2)$ is on $y = x^2/8$; E: $(6, 3)$ on $y = x^2/12$.

Table of Function Families and Their Isoderivative Curves

(The curve whose equation is in Column B consists of all points (x, y) where any family member from Column A has a derivative equal to V.)

	Column A -- Function Family	Column B -- Isoderivative Curve
Type I.	$y = f(x) + C$	$x = (f')^{-1}(V)$, possibly multi-valued. vertical lines
Type II.	$y = C f(x)$	$y = V f(x)/f'(x)$, all x such that $f'(x) \neq 0$.
a.	$y = Cx^2$	$y = Vx/2$, $\left\{ x \neq 0.\right.$ Includes all x, if $V = 0$. $\}$
b.	$y = Cx^r$, $r \neq 0$ fixed	$y = Vx/r$.
c.	$y = C e^x$	$y = V$, horizontal line
d.	$y = C \operatorname{Ln} x$	$y = V x \operatorname{Ln} x$.
e, f.	$y = C \sin x$, $y = C \cos x$	$y = V \tan x$, $y = -V \cot x$.

Column A -- Function Family	Column B -- Isoderivative Curve
g. $y = C \tan x$	$y = \frac{V}{2} \sin 2x$, $x \neq \frac{\pi}{2} + n\pi$
h. $y = C \sec x$	$y = V \cot x$, $x \neq \frac{n\pi}{2}$
i. $y = C(x^3 - 3x)$ has zero derivative at $x = \pm 1$.	$y = \frac{Vx}{3}\left(1 - \frac{2}{x^2-1}\right)$

Type III. $y = f(Cx)$ — $V x = f^{-1}(y) \cdot f'(f^{-1}(y))$.

a. $y = \sin Cx$ — $V x = \left[\sin^{-1}(y)\right]\sqrt{1 - y^2}$.

b. $y = \tan Cx$ — $Vx = \left[\tan^{-1}(y)\right] \cdot (1 + y^2)$.

c. $y = e^{Cx}$ — $V x = y \operatorname{Ln} y$.

Type IV. Miscellaneous

a. $x^2 + y^2 = c^2$ — $y = -x/V$, $x \neq 0$. radial line

b. $x^2 - 2Cx + y^2 = 0$ circle tangent to y-axis, with center on x-axis at $(C, 0)$ — $y = x\left(V \pm \sqrt{V^2 + 1}\right)$, $x \neq 0$. Use both signs -- each I-curve is 2 perpendicular lines, minus origin.

c. $y = e^{Cx^2}$ — $V x = 2 y \operatorname{Ln} y$

d. $y = x^3/3 - C^2 x$, derivative is zero at $x = \pm C$. — $y = -2x^3/3 + V x$

e. $y = x^C$ — $y \operatorname{Ln} y = V x \operatorname{Ln} x$.

f. $y^2 + x^2/C^2 = 1$ ellipse with y-intercepts ± 1. — $y = Vx/2 \pm \frac{1}{2}\sqrt{V^2 x^2 + 4}$, $x \neq 0$. Choose sign so 1st and 2nd terms have opposite signs. Then $y \to 0$ as $|x| \to \infty$.

Inverse Problem Examples.

a. $y = Ce^{\int \frac{dx}{g(x)}}$ — $y = V g(x)$

b. $y = C e^{\sin x}$ — $y = V \sec x$

c. $y = C e^{-1/x}$ — $y = V x^2$, $x \neq 0$.

d. $y = Ce^{x^{1-r}/1-r}$ — $y = Vx^r$, $x \neq 0$, $r \neq 1$ fixed

e. $y = Ce^{-e^{-x}}$ — $y = Ve^x$