Differentiable Integrals and Discontinuous Integrands

by

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One version of the Fundamental Theorem of Calculus is:

<u>Theorem</u>: If f is Riemann integrable on $I = [a, b]$, if $F(x) = \int_a^x f(t)\,dt$, and if f is continuous at the point $x_0 \in (a, b)$, then the derivative $F'(x_0)$ exists and equals $f(x_0)$.

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We investigate the differentiability of the integral F at a point x_0 where the integrand f is <u>not</u> continuous.

Our results are as follows:

(I.) If f has a <u>removable</u> discontinuity at x_0, i.e., $\lim_{x \to x_0} f(x) \neq f(x_0)$, then $F'(x_0)$ exists and equals $\lim_{x \to x_0} f(x)$. (Simply "redefine" f at x_0 as $f(x_0) = \lim_{x \to x_0} f(x)$ and apply the Theorem.)

(II.) If f has a <u>jump</u> discontinuity at x_0, i.e., $\lim_{x \to x_0-} f(x) \neq \lim_{x \to x_0+} f(x)$, then $F'(x_0)$ does <u>not</u> exist. However, F does have left- and right-hand derivatives at x_0, with $F'_-(x_0) = \lim_{x \to x_0-} f(x)$ and $F'_+(x_0) = \lim_{x \to x_0+} f(x)$. (The proof of this result is a one-sided modification of the proof of the Theorem found in standard texts.)

(III.) If f has an <u>essential</u> discontinuity (neither "removable" nor "jump") at x_0, then we cannot tell whether or not $F'(x_0)$ exists.

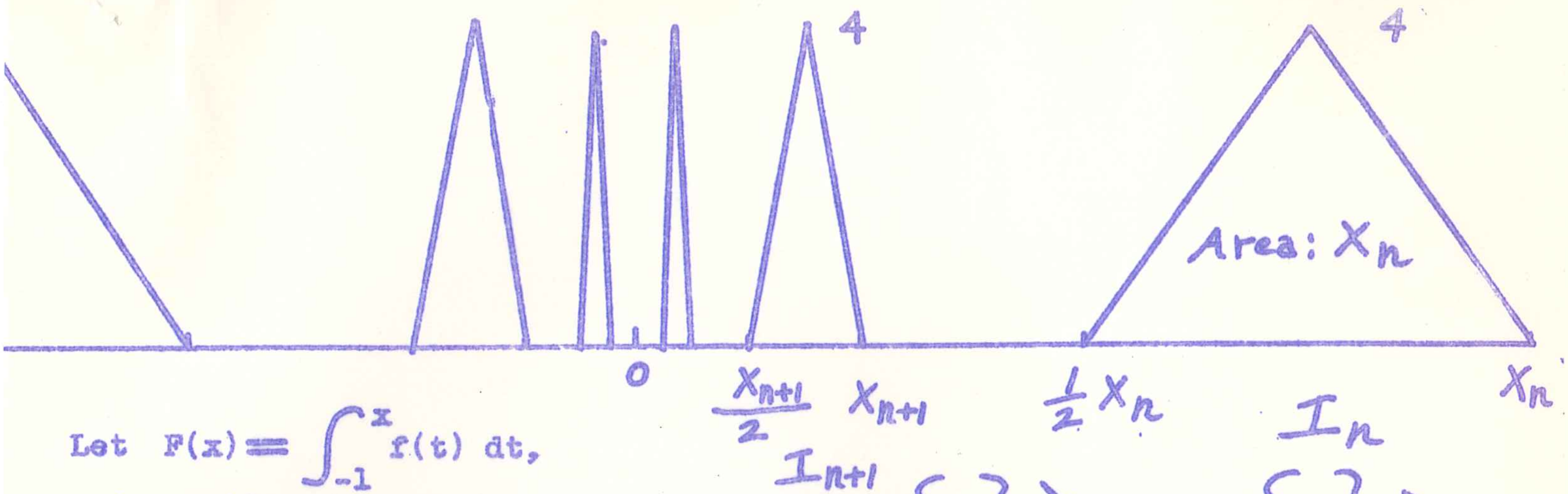
<u>Example A</u>: $f(x) = \sin \frac{1}{x}$, $x \neq 0$, $f(0) = 0$. $F(x) = \int_{-1}^{x} f(t)\,dt$. Here $F'(0) = 0$.

<u>Example B</u>: $f(x) = 1 + \sin \frac{1}{x}$, $x > 0$; $f(x) = -1 + \sin \frac{1}{x}$, $x < 0$; $f(0) = 0$. Here $F'(0)$ does not exist, but $F'_-(0) = -1$ and $F'_+(0) = 1$.

<u>Example C</u>: In this example, neither the right-hand nor the left-hand derivative of F exists at 0.

Let $x_n = 4^{-n}$, for $n = 0, 1, 2, \ldots$, and define f on the interval $I_n = [\frac{1}{2}x_n, x_n]$ so that the graph of f forms an isosceles triangle of height 4 (and area $x_n = 4^{-n}$).

Define $f = 0$ elsewhere on $[0, 1]$ and define f on $[-1, 0]$ so that its graph is symmetric with respect to the y-axis. (See diagram.)



Let $F(x) = \int_{-1}^{x} f(t)\,dt$,

and consider the two sequences of points $\{x_n\} \searrow 0$ and $\{y_n\} \searrow 0$, where $y_n = \frac{1}{2} x_n$. We can show that $\lim_{n \to \infty} \frac{F(x_n) - F(0)}{x_n - 0} = \frac{4}{3}$,

while $\lim_{n \to \infty} \frac{F(y_n) - F(0)}{y_n - 0} = \frac{2}{3}$, so that the right-hand derivative $F'_+(0)$ does not exist. Using the two sequences $\{-x_n\} \nearrow 0$ and $\{-y_n\} \nearrow 0$ demonstrates that $F'_-(0)$ also does not exist.

<u>Proof of Example A</u>: Let $F(x) = \int_{-1}^{x} \sin \frac{1}{t}\,dt$. We prove that:

(1) $$F'_+(0) = 0.$$

The proof that $F'_-(0) = 0$ is similar. Thus the derivative $F'(0)$ exists and equals 0.

By definition, $F'_+(0) = \lim_{h \to 0+} \frac{F(h) - F(0)}{h} =$

$\lim_{h \to 0+} \frac{1}{h} \int_0^h \sin \frac{1}{t}\,dt$.

Changing the variable of integration, $w = \frac{1}{t}$, we have:

(2) $$A_h = \frac{1}{h} \int_0^h \sin \frac{1}{t}\,dt = \frac{1}{h} \int_{\frac{1}{h}}^{\infty} \frac{\sin w}{w^2}\,dw.$$

If we divide the interval $\left[\frac{1}{h}, \infty\right)$ into subintervals $\left[k\pi, (k+1)\pi\right]$, we may represent the last integral in (2) as an alternating series. Because $\frac{1}{w^2}$ is a decreasing function of w, while $\sin w$ is periodic, this alternating series converges. In fact it converges absolutely.

We may show that

(3) $$|A_h| \leq 2\pi h,$$

making $\lim_{h \to 0+} A_h = 0$ and proving (1).