REPRESENTATIONS OF C^*-ALGEBRAS

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ABSTRACT

Given a C^*-algebra A, a fiber-bundle-like structure (B, π, Q) is constructed from A and its set Q of primitive ideals. The space of continuous cross-sections of this bundle is then given the structure of a C^*-algebra. The main result of the theory is a non-commutative analogue of the classical Gelfand-Naimark Theorem for commutative C^*-algebras.

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1. INTRODUCTION

In recent years considerable attention has been focused on the classification and structure theory of non-commutative C^*-algebras ([3], [4], [5], [6], [11], and [12]). The main purpose of this paper is to announce a non-commutative analogue of the classical Gelfand-Naimark theorem for commutative C^*-algebras.

Suppose that A is a C^*-algebra and let Q denote the Jacobson structure space of A. At each point I in Q consider the primitive C^*-algebra A/I, and let $B = \bigcup \{A/I: I \in Q\}$. For each x in A define a map $\hat{x}: Q \to B$ by $\hat{x}(I) = x + I \in A/I$. The underlying idea in the above papers is to construct a natural topology on the set B which makes each map $\hat{x}$ continuous, and then to represent A in the algebra of all such B-valued continuous functions. In order to define the topology on B various conditions are imposed on the C^*-algebra A or the topology on Q (or both). Usually these conditions are rather stringent. For example, in [12] it is assumed that each irreducible *-representation of A has the same finite dimension n, and in [11] it is assumed that the Jacobson topology is Hausdorff. In this paper a different point of view is adopted which allows us to (simultaneously) construct natural topologies on B and Q from A which makes each $\hat{x}$ continuous; moreover, these topologies are more or less independent of any particular properties of A. Then, the problem of representing A as an algebra of such continuous functions on Q is considered.

2. UNIFORM CT-BUNDLES.

This section will contain the topological foundations which are necessary for our representation theory.

An S-bundle (resp. T-bundle) is a triple (B, π, Q) where B and Q are non-empty sets (resp. topological spaces) and $\pi: B \to Q$ is a surjective (resp. continuous surjective) map. The map π is called the projection, and $B_q = \pi^{-1}(q)$ the fiber over the point q in Q. A local cross-section of a T-bundle over an open set U in Q is a continuous map $f: U \to B$ satisfying $\pi \circ f = 1_U$. The set of all local cross-sections over U will be denoted by $S(U, B)$.

2.1 DEFINITION. A CT-bundle is a T-bundle (B, π, Q) such that each point of B lies in the image of some local cross-section. A bundle uniformity on an S-bundle is a uniform structure Z on B restricted to the set $B \oplus B = \{(a, b): \pi(a) = \pi(b)\}$ in $B \times B$.

Let (B, π, Q) be a T-bundle and Z a bundle uniformity; if $b \in B$, $F \in Z$, and f is any local cross-section with domain D(f) satisfying $b \in \text{Image}(f)$, define $F[f] = \{a \in B: \pi(a) \in D(f) \text{ and } (f(\pi(a)), a) \in F\}$.

2.2 DEFINITION. Let (B, π, Q) be a CT-bundle, Z a bundle uniformity, and let S^* denote the set of all local cross-sections of B. The quadruple (B, π, Q, Z) will be called a uniform CT-bundle if for each element b in B a neighborhood base at b is given by the family $N_b = \{F[f]: F \in Z,\ f \in S^* \text{ with } b \in \text{Image}(f)\}$.

Suppose that (B, π, Q) is an S-bundle. A nonempty family M of maps from Q into B will be called accommodating if $\pi \circ f = 1_Q$ for each f in M, and each point of B lies in the image of some f in M.

Let (B, π, Q) be an S-bundle, Z a bundle uniformity on B, and M any accommodating family of maps. For each b in B consider the nonempty family $R_b = \{F[f]: F \in Z,\ f \in M \text{ with } b \in \text{Image}(f)\}$. The unique topology on B which has R_b as subbase at b will be called the <u>bundle topology</u>, and will be denoted by $t(B;M)$.

2.3 LEMMA. Let (B, π, Q) be an S-bundle, Z a bundle uniformity, and M any accommodating family of maps from Q into B. Assume that B is furnished with the bundle topology, and for each $f \in M$, let I_f denote the set Image(f) with the relative topology. For each $f \in M$ there is a unique topology $s(f;Q)$ on Q such that π restricted to I_f is a homeomorphism.

The individual topologies in the family $\{s(f:Q) : f \in M\}$ will be called the <u>component sectional topologies</u> on Q. We define the <u>sectional topology</u> on Q to be the least upper bound of the family of all component sectional topologies on Q, and denote it by $s(M;Q)$. It is the smallest topology on Q making each map in M continuous. Finally, the nonempty family of sets

$$D = \{V \cap \pi^{-1}(U): V \in t(B;M) \text{ and } U \in s(M;Q)\}$$

form a base for a topology $t(B)$ on B which contains the bundle topology; we call $t(B)$ the CT-<u>bundle topology.</u> It is immediate that $\pi : B \to Q$ is continuous relative to the topologies $t(B)$ and $s(M;Q)$.

The following theorem is important because it establishes the existence of suffiently many continuous cross-sections for our later representation theory.

2.4 THEOREM. (Uniform CT-Bundle Theorem) Let (B, π, Q) be an S-bundle, Z a bundle uniformity on B, and M any accommodating family of maps from Q into B. Furnish B with the CT-bundle topology t(B) and Q with the sectional topology s(M;Q). Then (B, π, Q, Z) is a uniform CT-bundle with $M \subset S(Q, B)$. In particular, every mapping in M is continuous.

The next proposition shows where t(B) and s(M;Q) stand in relation to other topologies on B and Q which make (B, π, Q, Z) into a uniform CT-bundle.

2.5 PROPOSITION. Let (B, π, Q) be an S-bundle, Z a bundle uniformity on B, and let M be any accommodating family of maps from Q into B. Assume that P(B) and P(Q) are any topologies on B and Q respectively such that (B, π, Q, Z) is a uniform CT-bundle with $M \subset S(Q, B)$. Then $t(B) \subset P(B)$ and $s(M;Q) \subset P(Q)$.

The following lemma is concerned with making the set of cross-sections of a uniform CT-bundle into a uniform space.

2.6 LEMMA. Let (B, π, Q, Z) be a uniform CT-bundle. For each entourage $F \in Z$ define a set F^+ in $S(Q, B) \times S(Q, B)$ by

$$F^+ = \{(f, g): (f(q), g(q)) \in F \text{ for all } q \in Q\}.$$

Then the collection of sets $W = \{F^+ : F \in Z\}$ forms a base for a uniform structure Z^+ on $S(Q, B)$.

A more or less standard completeness argument establishes the next result.

2.7 LEMMA. Let (B, π, Q, Z) be a uniform CT-bundle with the property that each fiber $B_q = \pi^{-1}(q)$ is complete with respect to Z. Then $(S(Q,B), Z^+)$ is a complete uniform space.

3. THE REPRESENTATION OF C^*-ALGEBRAS

This section will contain a statement of our non-commutative analogue of the classical commutative Gelfand-Naimark theorem.

3.1 LEMMA. Let A be a C^*-algebra and Q the set of primitive ideals in A. Consider the S-bundle (B, π, Q) where $B = \cup\{A/I : I \in Q\}$ and $\pi : B \to Q$ is defined by $\pi(x + I) = I$. Then there exists a bundle uniformity Z on B which, when restricted to the fibers A/I, coincides with the natural invariant uniformities.

An application of the preceding results establishes the next lemma.

3.2 LEMMA. Let A be a C^*-algebra and (B, π, Q), Z the S-bundle and bundle uniformity of Lemma 3.1. For each x in A define a mapping $x : Q \to B$ by $x(I) = x + I$, and let $\hat{A} = \{\hat{x} : x \in A\}$. Then there exist unique smallest topologies on B and Q making (B, π, Q, Z) into a uniform CT-bundle with $A \subset S(Q,B)$. In particular, π and each $\hat{x}$ are continuous.

The uniform CT-bundle constructed in Lemma 3.2 will be called the C^*-<u>bundle</u>. Let A be a C^*-algebra and (B, π, Q, Z) the C^*-bundle.

It is immediate from Lemmas 2.6 and 2.7 that $(S(Q,B), Z^{+})$ is a complete uniform space. Under the natural pointwise operations $S(Q,B)$ is a *-algebra, and the map $x \to \hat{x}$ is a uniformly continuous algebraic *-isomorphism of A onto $\hat{A} \subset S(Q,B)$. In order to make A into a C^*-algebra it is necessary to introduce a boundedness condition on the cross-sections. Following [1, Exercise 7, p. 210] we define an equivalence relation R on $S(Q,B)$ by: $f \, R \, g$ if and only if there is a positive integer k (depending on f and g) such that

$$(f(I), g(I)) \in F^{k}$$

for all $I \in Q$ and symmetric entourages $F \in Z$; then define the <u>bounded cross-sections</u> $S_b(Q,B)$ in $S(Q,B)$ to be those equivalent to the zero cross-section. Although $S_b(Q,B)$ is a complete topological group it is not necessarily closed under ring multiplication so is not an algebra. However, using known connectedness arguments from uniform space theory [1, p. 204] we can prove:

3.3 LEMMA. Let A be a C^*-algebra and (B, π, Q, Z) the C^*-bundle. If the topology on Q is compact, then $S_b(Q,B) = S(Q,B)$.

Hence, if Q is compact Lemma 3.3 implies that $S_b(Q,B)$ is an algebra. Then one can show that the sup norm $\|f\| = \sup\{\|f(I)\| : I \in Q\}$ on $S_b(Q,B)$ is finite, $S_b(Q,B)$ is a C^*-algebra, and $\hat{A} \subset S_b(Q,B)$. Since $x \to \hat{x}$ is an injective *-homomorphism from the C^*-algebra A into the C^*-algebra $S_b(Q,B)$ it is an isometry [3, Prop. (1.8.1), p. 16] and therefore has uniformly closed range $\hat{A}$ in $S_b(Q,B)$. Consequently, $\hat{A}$ is a C^*-algebra. Summarizing our results we can state:

3.4 THEOREM. (Gelfand-Naimark Theorem) Let A be a C^*-algebra, Q the set of primitive ideals in A, and (B, π, Q, Z) the C^*-bundle. If Q is compact, the map $x \to \hat{x}$ is an isometric *-isomorphism of A onto the C^*-algebra $\hat{A}$ in $S_b(Q, B)$. In particular, A is isometrically *-isomorphic with a closed subalgebra of $S_b(Q, B)$.

4. REMARKS

1. Theorem 3.4 raises two interesting questions: (a) If the C^*-algebra A has an identity, can the hypothesis of compactness on Q be removed? (b) When is the representation onto? That is, when is it true that $\hat{A} = S_b(Q, B)$? The author believes that (a) can be answered in the affirmative but we have no proof yet. Question (b) appears to be much more difficult. It is clear that a Stone-Weierstrass theorem of some kind is needed. The known theorems of this type for C^*-algebras ([7], [9], [11]) do not appear to be directly applicable.

2. The author has recently learned that similar results on locally convex topological algebras have been announced by K. H. Hofmann [8], and that proofs are to appear in a (joint) forthcoming A. M. S. Memoir by J. Dauns and K. H. Hofmann [2].

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