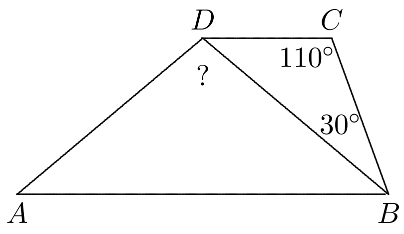
SOLUTIONS IOWA 2005

PROBLEM 1: How big is that angle?



It is 100°. From triangle BCD we see that angle BDC is 40°, and thus angle DBA is likewise 40°. But triangle ABD is isosceles, with angle DAB equal to angle DBA, which is 40°. Thus the angle ADB is 100°.

PROBLEM 2: From APs to squares.

(a) We may write the sum as the sum of two arithmetic progressions:

$$1 + 2 + 3 + \cdots + 2005 = \frac{2005 \cdot 2006}{2}$$

and

$$1 + 2 + 3 + \cdots + 2004 = \frac{2004 \cdot 2005}{2}.$$

Then the sum of the two is

$$\frac{2005(2004 + 2006)}{2} = 2005^2.$$

(b) In general,

$$\begin{aligned}
&[1 + 2 + 3 + \cdots + (n-1) + n] + [(n-1) + \cdots + 3 + 2 + 1] \\
&= \frac{n(n+1)}{2} + \frac{(n-1)n}{2} \\
&= \frac{n^2 + n + n^2 - n}{2} \\
&= n^2.
\end{aligned}$$

ALTERNATE SOLUTION

It is not necessary to have remembered a formula for the sum of an arithmetic progression. One may write the sum in (b)

$$\begin{array}{ccccccccc}
 & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + n \\
+ & (n-1) & + & (n-2) & + & (n-3) & + \cdots + & 1 & \\
= & n & + & n & + & n & + \cdots + & n & + n \\
 & & & & & = & & (n-1)n & + n \\
 & & & & & = & & n^2. &
\end{array}$$

PROBLEM 3: Mystery function.

The answers are: (a) $f^{(10)}(2) = 4$, and (b) $f(3) = 4e - 1$.

(a) From $f'(x) = 1 + f(x)$ we get $f''(x) = f'(x)$, and by an easy induction, $f^{(n)}(x) = f'(x)$ for all $n \geq 1$. Thus $f^{(10)}(x) = f'(x) = 1 + f(x)$, and $f^{(10)}(2) = 1 + f(2) = 4$.

(b) From $f'(x) = 1 + f(x)$ we have $\frac{dy}{1+y} = dx$, so $\ln(1 + y) = x + C$ for some constant C. Since $y = 3$ when $x = 2$ we have $C = \ln 4 - 2$, and

$$x - 2 = \ln(1 + y) - \ln 4 = \ln \frac{1 + y}{4}.$$

Putting $x = 3$ gives us $\ln \frac{1+y}{4} = 1$, $\frac{1+y}{4} = e$, and $y = 4e - 1$.

PROBLEM 4: Find a winning strategy.

Player 2 has a winning strategy. Note that the player who moves to square 1999 wins, and the one who moves to 1993 can move next to 1999. In general the positions from which a win can be assured are numbers which, like 2005, are of the form $6n + 1$. When Player 1 advances the piece k squares from position $6n + 1$ (and this includes position 1), Player 2 moves it $6 - k$ squares to $6n + 6 = 6(n + 1) + 1$. (Note that if the final number were of the form $6n+r$ with $r = 2, 3, 4$ or 5, the first player could win by moving to square r, or if $r = 0$, to square 6.)

PROBLEM 5: Greatest common divisor.

The answer is $M = 48$. Since $f(1) = -48$, it is clear that M must be a divisor of 48. Using the factorization $u^n - 1 = (u - 1)(u^{n-1} + u^{n-2} + \cdots + u + 1)$ we may write

$$\begin{aligned} f(n) &= 25^n - 1 - 72n \\ &= 24(25^{n-1} + 25^{n-2} + \cdots + 25 + 1) - 24(3n). \end{aligned}$$

Whether n is even or odd, this is an even multiple of 24, so is divisible by 48. Thus, $M = 48$ is the largest common divisor of all the values of $f(n)$.

ALTERNATE SOLUTION

As above we see from $f(1)$ that M must be a divisor of 48. Now examine $f(n) = 25^n - 72n - 1$ modulo 3 and modulo 16 separately: Modulo 3, $f(n) \equiv 1^n - 0 - 1 = 0$, and modulo 16, $25 \equiv 9$ and $25^2 \equiv 9^2 \equiv 1$, while $72 \equiv 8$. Thus, for even $n = 2m$, $f(2m) \equiv 1^m - 8(2m) - 1 = 1 - 0 - 1 = 0$, and for odd $n = 2m + 1$, $f(2m + 1) \equiv 25^{2m}25 \equiv 1^{2m}(9) - 8(2m + 1) - 1 \equiv 9 - 8 - 1 = 0$. Hence $f(n)$ is a multiple of $3 \cdot 16 = 48$ for all n, and therefore $M = 48$.

PROBLEM 6: Rational solutions.

It suffices to show that there are infinitely many rational pairs (x, y) with $y^2 - x^2 = a$ and $y > 0$. Thus we want $a = (y - x)(y + x)$. Let b be any nonzero rational number. We will find rational x and y such that $y - x = b$ and $y + x = a/b$. Indeed, the solution of this pair of equations is $x = (a - b^2)/2b$, $y = (a + b^2)/2b$, and both x and y are rational. Moveover, different values of b yield different values of $y - x$, and therefore different pairs (x, y). It remains only to insure that $y > 0$. For this it suffices that $b > 0$ and $b^2 > -a$, and it is clear that there are infinitely many rational values of b satisfying these conditions.

PROBLEM 7: Fractional part equation.

They are $x = \frac{-3+\sqrt{9+12n}}{6}$, where $n \in \{0, 1, 2, 3, 4, 5\}$. A necessary condition is that $0 \le x^3 < 1$, and therefore $0 \le x < 1$, because $0 \le \{u\} < 1$ for all real u. So we restrict attention to x with $0 \le x < 1$. Then

$$\{(x + 1)^3\} = \{x^3 + 3x^2 + 3x + 1\} = x^3 \iff 3x^2 + 3x = n,$$

where n is an integer and $0 \le x < 1$. The nonnegative root of this quadratic is

$$x = \frac{-3 + \sqrt{9 + 12n}}{6},$$

and this is real and lies in the interval $[0, 1)$ if and only if $0 \le n \le 5$.

PROBLEM 8: Limit of integrals.

For $x \le t \le 2x$ we have

$$0 < \frac{1}{\sqrt{t^3 + 4}} \le \frac{1}{\sqrt{x^3 + 4}},$$

and therefore

$$\begin{aligned} 0 &< \int_x^{2x} \frac{dt}{\sqrt{t^3 + 4}} \\ &\le \int_x^{2x} \frac{dt}{\sqrt{x^3 + 4}} \\ &= \frac{1}{\sqrt{x^3 + 4}} \int_x^{2x} dt \\ &= \frac{x}{\sqrt{x^3 + 4}}. \end{aligned}$$

But

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^3 + 4}} = 0,$$

so

$$\lim_{x \to \infty} \int_x^{2x} \frac{dt}{\sqrt{t^3 + 4}} = 0.$$

PROBLEM 9: Find the sum of squares.

Think complex numbers! From the equation

$$(a+bi)^3 = (a^3 - 3ab^2) + (3a^2b - b^3)i = 41 - 18i$$

we see that

$$a^2 + b^2 = |a+bi|^2 = |41 - 18i|^{\frac{2}{3}} = \left(\sqrt{2005}\right)^{\frac{2}{3}} = \sqrt[3]{2005}.$$

PROBLEM 10: A special sum of squares.

One such solution is $x_i = 1$ for $1 \leq i \leq 29$, $x_{30} = 929$, $x_{31} = 28769$. Here is one way to find solutions. One notices that the equation is satisfied if all $x_i = 1$. Put all but two of the x_i equal to 1, and see what we can do with the other two, which we call a and b. The condition is then

$$a^2 + b^2 + 29 = 31ab. \tag{1}$$

For fixed b, the condition (1) is that a be a root of the quadratic equation

$$x^2 - 31bx + (b^2 + 29) = 0. \tag{2}$$

The sum of the roots of this quadratic is $31b$. That a number pair (a, b) satisfies (1) means that $x = a$ satisfies (2), and therefore so does $x = 31b - a$. Thus, if the number pair (a, b) satisfies (1), then so does the number pair (a', b), where $a' = 31b - a$. Also, we may interchange the roles of a and b in this argument to conclude that if (a, b) satisfies (1), so does the pair (a, b'), with $b' = 31a - b$. Thus from the solution $(1, 1)$ we get another solution $(30, 1)$, and from it, $(30, 929)$, and again $(28769, 929)$.