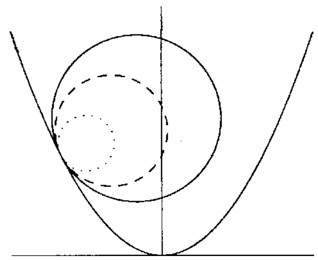
13th ANNUAL IOWA COLLEGIATE MATHEMATICS COMPETITION

Grinnell College, Saturday, March 10, 2007

(Problem set by Loren C. Larsen)

1. A circle rolls (or slides) along the inside arc of the parabola $y = x^2$. What is the radius of the largest circle that will eventually reach the bottom of the parabola (tangent at the origin) without getting stuck before getting there?



2. Arrange numbers in an infinite array of three columns as defined recursively in the following manner. The first row is $[1,\ 2,\ 4]$, and for $n > 1$, row n is $[a,\ b,\ a+b+1]$, where a and b, with $a < b$, are the two smallest positive integers that have not yet appeared as entries in rows $1, 2, \ldots, n-1$. The first two rows of the array are

	Column 1	Column 2	Column3
Row 1	1	2	4
Row 2	3	5	9

Note that after row 1 is given, 3 and 5 are the two smallest positive integers that have not yet been placed, so they appear in columns 1 and 2 of row 2. In which row and column will each of the following numbers appear?

(a) 2007

(b) 2008

(c) 2009

Provide rationale for your answers (a formal proof is not required).

3. Consider the set of all isosceles triangles (two equal sides) whose base is on the non-negative x-axis (that is, $x \geq 0$) and whose (third) vertex is on the curve $y = x(4-x)^3$, $0 \leq x \leq 4$ (an example of such a triangle is the one with vertices $(1,0)$, $(3,3)$, and $(5,0)$). Among these triangles, which one has the largest area, and what is it?

4. Define a sequence of *positive* numbers by $a_1 = a_2 = 1$ and for $n \geq 2$,

$$(a_{n+1})^2 = 1 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}}\right) + \frac{1}{a_1{}^2} + \frac{1}{a_2{}^2} + \cdots + \frac{1}{(a_{n-1})^2}.$$

Prove that a_n is a rational number for all n.

5. Squares are constructed outward on the respective sides of a parallelogram. Prove that the centers of these four squares are the vertices of a square.

6. For any two numbers a and b in the open interval $(-1, 1)$, let $\oplus$ be the binary operation defined by $a \oplus b = \dfrac{a+b}{1+ab}$. For example, $1/2 \oplus 1/2 = 1/(1 + 1/4) = 4/5$. These numbers form a group under this operation. For this problem, all you need to know is that the operation is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all a, b, c in $(-1, 1)$. Here's the problem: For an arbitrary a in $(-1, 1)$, find a closed algebraic formula for $\underbrace{a \oplus a \oplus \cdots \oplus a}_{k}$. (Note: By a "closed" formula we mean that, for example, an "open" sum, such as $2+4+\cdots+2n$, needs to be replaced by the "closed" formula $n(n+1)$.)

7. A solid ball of radius 1 is inside, and tangent to, a hollow sphere of radius 3. A light at the center of the sphere casts a shadow of the ball onto the sphere. Find the surface area of the shadowed region (on the encompassing sphere).

8. Take a positive integer n_0, and add to it the number of odd digits and subtract the number of even digits. This gives a new number n_1. Now repeat this procedure starting with n_1 to get n_2, then continue with n_2 to get n_3, and so on. For example: $8 \to 7 \to 8 \to 7 \to \cdots$ is a cycle of length 2. Similarly, $11 \to 13 \to 15 \to 17 \to 19 \to 21 \to 21 \to \cdots$ ends as a cycle of length 1 (a fixed point), and $996 \to 997 \to 1000 \to 998 \to 999 \to 1002 \to 1000 \to \cdots$ ends in a cycle of length 3. Will every starting number eventually end in a cycle? Prove or disprove.

9. Take a permutation of the numbers 1 to 5 and consider the following procedure for sorting them into increasing order. Pick any number that's out of place, and wedge it to its "proper" position, shifting others over to make room for it. Repeat this procedure as long as there are numbers that are out of place. For example, take 3 5 1 2 4, and choose the underlined out-of-place number at each step.

$$\begin{array}{ccccc}
3 & 5 & 1 & 2 & \underline{4} \\
3 & 5 & 1 & 4 & \underline{2} \\
\underline{3} & 2 & 5 & 1 & 4 \\
2 & \underline{5} & 3 & 1 & 4 \\
2 & \underline{3} & 1 & 4 & 5 \\
\underline{2} & 1 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array}$$

In this case, it has taken 6 steps to sort the numbers into their proper order. It might have taken no more than 5 steps, had we considered the numbers in order $1, 2, 3, 4, 5$. On the other hand, it might have taken more steps.

(a) Give an example of a permutation of $1, 2, 3, 4, 5$ that might require as many as 15 steps to sort using this procedure. Your example should give the sequence of steps by underlining the out-of-place numbers chosen for each step.

(b) It turns out that this sorting procedure will always terminate in 15 steps or less. Prove that there are an *even* number of permutations that may take as many as 15 steps to sort with this algorithm.

10. For each positive integer n, let

$$N(n) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n}{8} \right\rceil + \cdots + \left\lceil \frac{n}{2^k} \right\rceil,$$

where k is the unique integer such that $2^{k-1} \leq n < 2^k$, and $\lceil x \rceil$ denotes the smallest number greater than or equal to x. For example, $N(8) = \left\lceil \frac{8}{2} \right\rceil + \left\lceil \frac{8}{4} \right\rceil + \left\lceil \frac{8}{8} \right\rceil + \left\lceil \frac{8}{16} \right\rceil = 4 + 2 + 1 + 1 = 8$ and $N(10) = \left\lceil \frac{10}{2} \right\rceil + \left\lceil \frac{10}{4} \right\rceil + \left\lceil \frac{10}{8} \right\rceil + \left\lceil \frac{10}{16} \right\rceil = 5 + 3 + 2 + 1 = 11$.

(a) For which numbers n is $N(n) = n$?

(b) Prove that your characterization in part (a) is correct.