Twelfth Annual Iowa Collegiate Mathematics Competition

Luther College — March 4, 2006

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1. Twelfth Anniversary Question!

   Prove that $\displaystyle\int_0^1 x^{304}(1-x)^{2006}\,dx = \frac{304!\,2006!}{2311!}$.

2. No Fractions, Please

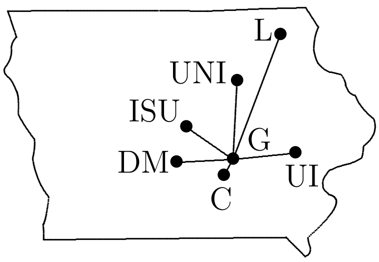
   Let $p(x)$ be a polynomial with rational coefficients. Prove that there is an integer k such that $p(x+k) - p(x)$ is a polynomial with integer coefficients.

3. Maximal Distance to $y = x^p$

   For what positive real numbers p is the maximal distance from the point $(1, 0)$ to the curve $y = x^p$, $0 \le x \le 1$, equal to 1?

4. Corny Random Exchange Problem

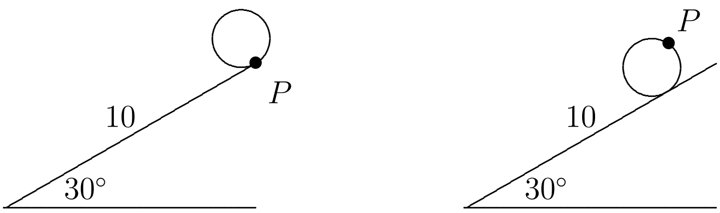
   Seven math majors representing each of the hosts and winners of the Iowa Collegiate Mathematics Contest decide to participate in an exchange program. The seven students represent Iowa State University, Central College, Luther College, the University of Northern Iowa, Grinnell College, the University of Iowa, and the Des Moines Area Community Colleges (see map below).



   First, the student at Grinnell randomly exchanges places with one of the other six students, with each equally likely to be chosen. The student now at Grinnell (who at this point is not originally from Grinnell) then exchanges places with one of the other six students, each again equally likely to be chosen. In all, there are six random exchanges of whichever student is currently at Grinnell and one of the other six students. What is the probability that each student ends up at his or her original location? Express your answer as a fraction in the form $m/6^r$.

5. Rolling Down a Ramp

   A circle of radius 1 rolls down a straight ramp of length 10 which has angle of elevation 30°. If the point P on the circle is initially tangent to the ramp, find the maximum height off the ground of the point P.



6. Factorial Divisibility

   Prove that $(n^2)!$ is divisible by $(n!)^{n+1}$ for all positive integers n.

7. Domino Tiling Game

   Consider the following game played on a 5×5 checkerboard. In turn, each player places a 1×2 domino so that it exactly covers two squares of the checkerboard. The last player able to place a domino wins. What are the shortest and longest possible games?

8. Integer Points on a Folium of Descartes

   Find *all* pairs of integers (x, y) that satisfy $x^3 + y^3 = 6xy$.

9. A Functional Equation

   Find all differentiable functions f for which

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))$$

   for all real numbers x and y.

10. Unsolvable

   Let $y = y(t)$ be a solution to the differential equation $y' + 2ty = t^2$. Evaluate

$$\lim_{t \to \infty} \frac{y}{t}.$$