Eleventh Annual
Iowa Collegiate Mathematics Competition
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To receive full credit, all problems, unless otherwise stated, require complete justification.

PROBLEM 1. How big is that angle?

In the trapezoid at the right, the sides $AB$ and $CD$ are parallel, and the diagonal $BD$ is equal in length to side $AD$. Given that angle $CBD$ measures $30^\circ$ and angle $DCB$ measures $110^\circ$, find the measure of angle $ADB$.

PROBLEM 2. From APs to Squares.

(a) Show that the sum

$$1 + 2 + 3 + \cdots + 2004 + 2005 + 2004 + \cdots + 3 + 2 + 1$$

is the square of an integer.

(b) Generalize the result in (a), with proof.

PROBLEM 3. Mystery function.

Suppose that the function $f$ satisfies $f'(x) = 1 + f(x)$ for all $x$. If $f(2) = 3$, find:

(a) $f^{(10)}(2)$ (where $f^{(10)}$ denotes the 10th derivative of $f$);
(b) $f(3)$.

Justify your answers.

PROBLEM 4. Find a winning strategy.

A game board consists of a linear path of 2005 squares, numbered from 1 to 2005. A game piece is initially on square 1, and two players alternately move it. On each move a player advances the piece by 1, 2, 3, 4 or 5 squares. Thus, the first player advances the piece to square 2, 3, 4, 5 or 6. The player who moves onto square 2005 wins. Describe a winning strategy for one of the players, and make clear that it wins.
PROBLEM 5. Greatest common divisor.

Let \( f(n) = 25n - 72n - 1 \). Determine, with proof, the largest integer \( M \) such that \( f(n) \) is divisible by \( M \) for every positive integer \( n \).

PROBLEM 6. Rational solutions.

Prove that for every rational number \( a \), the equation
\[
y = \sqrt{x^2 + a}
\]
has infinitely many solutions \((x, y)\) with \( x \) and \( y \) rational.

PROBLEM 7. Fractional part equation.

For real numbers \( u \), let \( \{u\} = u - \lfloor u \rfloor \) denote the fractional part of \( u \). Here \( \lfloor u \rfloor \) denotes, as usual, the greatest integer less than or equal to \( u \). For example, \( \{\pi\} = \pi - 3 \), and \( \{-2.4\} = -2.4 - (-3) = 0.6 \). Find all real \( x \) such that
\[
\{(x + 1)^3\} = x^3.
\]

PROBLEM 8. Limit of integrals.

Evaluate
\[
\lim_{x \to \infty} \int_x^{2x} \frac{dt}{\sqrt{t^3 + 4}},
\]
and justify your answer.

PROBLEM 9. Find the sum of squares.

Given that \( a \) and \( b \) are real numbers satisfying \( a^3 - 3ab^2 = 41 \) and \( b^3 - 3a^2b = 18 \), determine \( a^2 + b^2 \).

(Hint: Think complex numbers!)

PROBLEM 10. A special sum of squares.

Find positive integers \( x_1, x_2, \ldots, x_{31} \), at least one of which is greater than 2005, satisfying
\[
x_1^2 + x_2^2 + \cdots + x_{31}^2 = 31x_1x_2 \cdots x_{31}.
\]