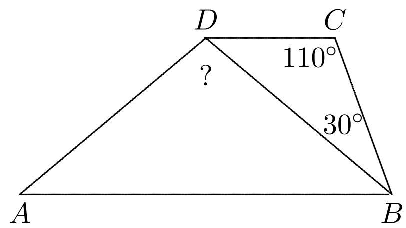
**Eleventh Annual**
**Iowa Collegiate Mathematics Competition**
Iowa State University April 16, 2005

Problems by Jerry Heuer, Concordia College (heuer@cord.edu)

To receive full credit, all problems, unless otherwise stated, require complete justification.

## PROBLEM 1. How big is that angle?

In the trapezoid at the right, the sides $AB$ and $CD$ are parallel, and the diagonal $BD$ is equal in length to side $AD$. Given that angle $CBD$ measures $30°$ and angle $DCB$ measures $110°$, find the measure of angle $ADB$.




## PROBLEM 2. From APs to Squares.

(a) Show that the sum

$$1 + 2 + 3 + \cdots + 2004 + 2005 + 2004 + \cdots + 3 + 2 + 1$$

is the square of an integer.

(b) Generalize the result in (a), with proof.

## PROBLEM 3. Mystery function.

Suppose that the function $f$ satisfies $f'(x) = 1 + f(x)$ for all $x$. If $f(2) = 3$, find:
(a) $f^{(10)}(2)$ (where $f^{(10)}$ denotes the 10th derivative of $f$);
(b) $f(3)$.
Justify your answers.

## PROBLEM 4. Find a winning strategy.

A game board consists of a linear path of 2005 squares, numbered from 1 to 2005. A game piece is initially on square 1, and two players alternately move it. On each move a player advances the piece by 1, 2, 3, 4 or 5 squares. Thus, the first player advances the piece to square 2, 3, 4, 5 or 6. The player who moves onto square 2005 wins. Describe a winning strategy for one of the players, and make clear that it wins.

**PROBLEM 5. Greatest common divisor.**

Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer $M$ such that $f(n)$ is divisible by $M$ for every positive integer $n$.

**PROBLEM 6. Rational solutions.**

Prove that for every rational number $a$, the equation

$$y = \sqrt{x^2 + a}$$

has infinitely many solutions $(x, y)$ with $x$ and $y$ rational.

**PROBLEM 7. Fractional part equation.**

For real numbers $u$, let $\{u\} = u - \lfloor u \rfloor$ denote the fractional part of $u$. Here $\lfloor u \rfloor$ denotes, as usual, the greatest integer less than or equal to $u$. For example, $\{\pi\} = \pi - 3$, and $\{-2.4\} = -2.4 - (-3) = 0.6$. Find all real $x$ such that

$$\{(x + 1)^3\} = x^3.$$

**PROBLEM 8. Limit of integrals.**

Evaluate

$$\lim_{x \to \infty} \int_x^{2x} \frac{dt}{\sqrt{t^3 + 4}},$$

and justify your answer.

**PROBLEM 9. Find the sum of squares.**

Given that $a$ and $b$ are real numbers satisfying $a^3 - 3ab^2 = 41$ and $b^3 - 3a^2b = 18$, determine $a^2 + b^2$.

(Hint: Think complex numbers!)

**PROBLEM 10. A special sum of squares.**

Find positive integers $x_1, x_2, \ldots, x_{31}$, at least one of which is greater than 2005, satisfying

$$x_1^2 + x_2^2 + \cdots + x_{31}^2 = 31 x_1 x_2 \cdots x_{31}.$$