Tenth Annual
Iowa Collegiate Mathematics Competition

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Problems

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The problems below are listed in roughly increasing order of difficulty. Each problem requires proof or justification. An answer alone is not sufficient. Calculators are permitted, but not particularly useful.

1 A Darwinian Struggle

At time $t = 0$ minutes, a virus is placed into a colony of 2,004 bacteria. Every minute, each virus destroys one bacterium apiece, after which all the bacteria and viruses divide in two. For example, at $t = 1$, there will be $2003 \times 2 = 4006$ bacteria and 2 viruses. At $t = 2$, there will be 4004×2 bacteria and 4 viruses, etc. Will the bacteria be driven to extinction? If so, when will this happen?

2 Slicing a Sphere

A sphere is cut by parallel planes into n pieces. It turns out that the total surface area of these pieces is exactly 2004 times the area of the sphere. What is the smallest possible value for n?

3 A Finite Pattern

Let $P(x)$ be a 2004th-degree polynomial with real coefficients, satisfying $P(n) = n$, for $n = 1, 2, 3, \ldots, 2004$. Prove that it is possible for $P(2005)$ to equal any real value *except* 2005.

4 The First Derangement

One can list the $n!$ permutations of $\{1, 2, \ldots, n\}$ in "ascending" order, treating each permutation as a number written base-$(n+1)$. For example, if $n = 3$, the permutations, in order, are

$$123 < 132 < 213 < 231 < 312 < 321.$$

A permutation is called a *derangement* if no number is in its "correct" (original) position. For example, if $n = 3$, the only derangements are 231 and 312, since 123 leaves all 3 numbers in correct order, 132 leaves 1 unchanged, 213 leaves 3 unchanged, and 321 leaves 2 unchanged.

For $n \geq 2$, let a_n denote the rank of the first derangement in the list of the permutations of $\{1, 2, \ldots, n\}$ in numerical order. For example, $a_3 = 4$, since 231 was the 4th permutation in the list above.

What is the rightmost (unit's) digit of a_{2004}?

5 Tiling the Plane

It is obvious that one can tile the plane with infinitely many unit squares. (A unit square is a square with side length 1, and by *tile* we mean placing the squares so that there is no overlap, every vertex of one square coincides with the vertex of another square, and no point in the plane is left uncovered.)

Likewise, it is obvious that one can tile the plane with infinitely many unit equilateral triangles. But what if we use both? Which of the following sets can tile the plane?

(a) Infinitely many unit squares and infinitely many unit equilateral triangles.

(b) Infinitely many unit squares and finitely many (and at least one) unit equilateral triangles.

(c) Finitely many unit squares (at least one) and infinitely many unit equilateral triangles.

6 Arts and Crafts

Given two polyhedra, all of whose edges have length 1: a pyramid with a square base, and a tetrahedron. Suppose we glue the two polyhedra together along a triangular face (so that the attached faces exactly overlap). How many faces does the new solid have?

7 Circles and Pentagons

Let $ABCDE$ be a pentagon (not necessarily regular) inscribed in a circle. Let A' be the midpoint of CD, B' the midpoint of DE, C' the midpoint of EA, D' the midpoint of AB, and E' the midpoint of BC. Show that the midpoints of $A'C'$, $C'E'$, $E'B'$, $B'D'$ and $D'A'$ all lie on a circle.

8 Acute and Chronic Velocity

A highway patrolman pulled over a math student, and said, "I've been following you for 3 hours. At 3 PM, you were at mile-marker 100, and now you are at mile-marker 310. This proves that you have been traveling at 70 MPH, for at least an instant of time during the past 3 hours."

The student says, "I suppose you have assumed that my position along the highway (my mile-marker location) is a differentiable function of time, and you are attempting to use the Mean Value Theorem. Officer, not to be disrespectful, but how can you be sure that my position function is differentiable? What if it is merely continuous?" The officer responds, "OK, smart aleck, then I can prove that there was a period of exactly one hour during the past 3 hours, during which you traveled a net distance of exactly 70 miles." Explain his reasoning.

9 Even Chances

Pick a number randomly from the first 2^{2004} rows of Pascal's Triangle. To the nearest percent, what is the probability that this number is even?

10 An Irrational Conclusion

Find all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions.

1. $f(0) = 2004$.
2. If x is irrational, then $f(x)$ is irrational.

(A polynomial is *monic* if its highest-degree term has coefficient 1.)