# Ninth Annual Iowa Intercollegiate Mathematics Competition

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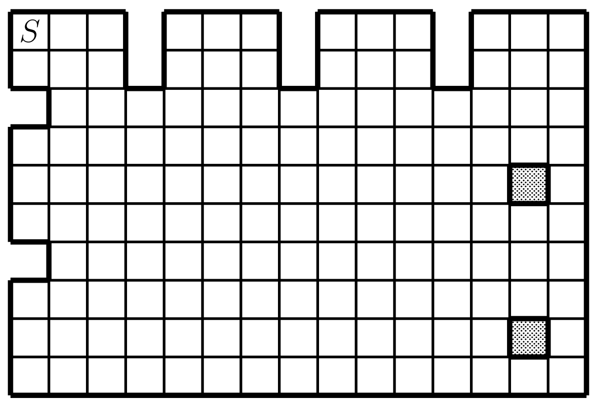
Problems set by Loren Larson of Northfield, Minnesota (lllarsson@earthlink.net).

To receive full credit, all problems, unless otherwise stated, require complete justification.

**1. Queen Sophie's tour** Sophie, queen bee, discovers an abnormal layer of cubical cells in her hive (see figure; the two darkened cells are damaged and unavailable). Sophie intends to lay eggs in each cell, and with great insight and intuition, she starts at $S$ and moves in single steps from one cell to another sharing a common side. When she returns to $S$ (after 140 steps), she has visited each available cell exactly once. Can you piece together a map of her tour? (Justification not required; a sketch of the tour will suffice.)




**2. Polynomial search** Can you find (that is, describe, characterize, or identify in some way) all non-zero polynomials $P(x)$ for which $P(x) = P'(x)P''(x)$ for all $x$?

**3. Superstar versus supercomputer** Neither *Mathematica* nor *Maple* can find the exact value of the following definite integral. Can you?

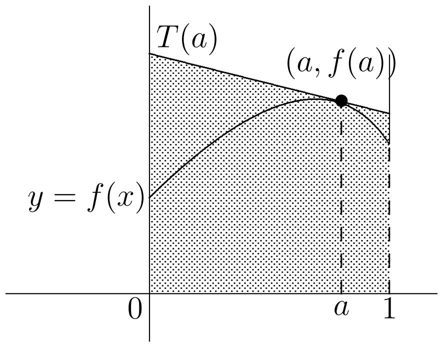
$$\int_0^2 (3x^2 - 3x + 1)\cos(x^3 - 3x^2 + 4x - 2)\, dx$$

**4. Periodic points** Let $S$ be a set (possibly infinite) and $G$ a function from $S$ to itself. Let $G_n$ be the $n$th iterate of $G$; for example, $G_3(s) = G(G(G(s)))$. Suppose that for each $s$ in $S$ there exists a positive integer $n$, depending on $s$, such that $G_n(s) = s$.

**a.** Is $G$ onto $S$? (That is, for each $s$ in $S$, is there a $t$ in $S$ such that $G(t) = s$?)

**b.** Is $G$ one-to-one? (That is, if $G(s) = G(t)$, must $s = t$?)

**5. Minimal enclosure** Suppose that $f$ has two derivatives on the interval $[0,1]$ and that $f(x) > 0$ and $f''(x) < 0$ ($f$ is concave down) for all $x$ in this interval. Let $T(a)$ denote the tangent to $y = f(x)$ at the point $(a, f(a))$. Can you determine the value of $a$ (it may depend on $f$) that minimizes the area $A(a)$ in the region (shaded) under the tangent (above the $x$-axis and between $x = 0$ and $x = 1$)?




**6. Dice game** Jack and Jill alternately throw a pair of dice; Jack starts (and Jill comes tumbling after). Jack wins if the sum of the two dice totals 6 points before Jill rolls a sum of 7 points, in which case she wins. Can you determine who has the better chance of winning, and by how much? (That is, find Jack's and Jill's probability of winning.)

**7. Representations with squares**

$$\begin{aligned} 1 &= 1^2 \\ 2 &= -1^2 - 2^2 - 3^2 + 4^2 \\ 3 &= -1^2 + 2^2 \\ 4 &= -1^2 - 2^2 + 3^2 \\ 5 &= 1^2 + 2^2 \end{aligned}$$

**a.** Can every number be expressed as the sum of the first so many squares, if appropriate signs are allotted?

**b.** Can you write 59 as a sum of the first *ten* positive squares with some choice of plus/minus signs? If so, in how many different ways can you do it? (A justification for the complete list is not required.)

**8. Crossnumber puzzle** Each row and column of the grid represents a three digit number in base 10. Can you find nine digits (not necessarily different) which simultaneously satisfy the stated conditions? (No need to justify answers in this problem.)

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | | |
| 5 | | |

**Across**

**1.** Divisible by 7.

**4.** Uniquely expressible as a sum of two non-zero squares.

**5.** A prime number.

**Down**

**1.** Either a square or a cube.

**2.** Has more divisors than any other 3-digit decimal number.

**3.** When expressed in base 2, or in base 3, it has exactly two non-zero "digits."

**9. Extrapolation par-excellence** For $n = 0, 1, 2, \ldots,$ let $F_n = x^n \ln x, x > 0$, and set $A_n = \dfrac{F_n^{(n)}(1)}{n!}$ (where $F_n^{(n)}$ denotes the $n$th derivative of $F_n$). To what value, if any, does the infinite sequence $A_0, A_1, A_2, A_3, \ldots$ converge?

**10. An integer sequence?** For $n = 0, 1, 2, \ldots$, let

$$a_n = 2^{n/2+1} \frac{\sin(nt)}{\sqrt{7}}, \qquad \text{where} \qquad t = \arctan \sqrt{7}$$

Is $a_n$ an integer for all (nonnegative integers) $n$?