27 $^{\text{th}}$ Annual Iowa Collegiate Mathematics Competition—Solutions

1. **Marbelous!** A bag contains some marbles, each of which is either red or blue. If one red marble is removed, then $\frac{1}{7}$ of the remaining marbles are red. If a red marble is not removed but instead two blue marbles are removed, then $\frac{1}{5}$ of the remaining marbles are red. How many marbles were in the bag originally? Be sure to justify your answer.

   Solution 1. There are 22 marbles in the bag originally. Let M be the total number of marbles in the bag and let R be the number of red marbles in the bag. The two conditions lead to the equations

   $$\frac{R-1}{M-1} = \frac{1}{7} \qquad \text{and} \qquad \frac{R}{M-2} = \frac{1}{5}.$$

   From this we obtain the system of equations

   $$\begin{aligned} 7R - M &= 6 \\ 5R - M &= -2. \end{aligned}$$

   On solving, we find $R = 4$ and $M = 22$.

2. **Goldbach Lite.** Prove that every positive integer greater than 6 can be written as a sum of two relatively prime integers, both of which are greater than 1. (Two positive integers are relatively prime if the only positive integer that is a factor of both numbers is 1.)

   Solution 2. If $n = 2k+1 \geq 7$ is odd, then , then $n = k+(k+1)$ expresses n as a sum of relatively prime positive integers greater than 1. (And of course $n = (n-2)+2$ also gives an allowable decomposition.)

   If $n = 4k \geq 8$ is a multiple of 4, then $n = (2k-1)+(2k+1)$ exhibits a decomposition of the desired form.

   If $n = 4k+2 \geq 10$, then $n = (2k-1)+(2k+3)$ expresses n as a sum of two odd numbers whose difference is 4. Any two such odd numbers are relatively prime.

3. **Stonework.** A two player game starts with a pile of 100 stones. For a turn a player selects one of the piles of stones and divides in into two nonempty piles, but with the provision that a pile with just two stones or one stone cannot be divided. The

winner is the last person to make a legal move. Which player has a winning strategy and what is one such strategy?

Solution 3. The first person has a winning strategy if the initial number of stones is even and greater than or equal to 4. Let the two players be A and B, with player A going first. If the starting pile has $2n$ stones for some integer $n \geq 2$, then player A can divide the pile into two piles with n stones each. For the sake of discussion, assume one pile of n stones is placed on a red table and the other pile of n stones is on a blue table. After this first move, each time player B makes a move, dividing a pile on the (red/blue) table, player A makes the same move but on the (blue/red) table. After player A moves, the pile configurations on the two tables are identical, so whenever player B can move, player A can also move. Therefore player A will be the last player to make a legal move.

Extra Challenge: Suppose the game starts with a pile of 101 stones. Which player has a winning strategy and what is it?

4. **Are You Feeling Lucky?** The game *Cant Stop*© is played with four standard six-sided dice. On a turn, a player rolls the four dice, then sum pairs of the dice. What is the probability that *at least* one pair of the four dice sums to seven?

Solution 4. The probability that at least one pair of dice sums to 7 is $\frac{139}{216}$. To justify this, we first find the probability that there is no pair of dice that sum to 7. In our argument we assume that the four dice are distinguishable, for example, the dice are of four different colors.

The pairs of die numbers that sum to 7 are

$$A = \{1, 6\}, \qquad B = \{2, 5\}, \qquad \text{and} \qquad C = \{3, 4\}.$$

Let a, b, c, d be the numbers on the four dice. The no pair of these numbers sums to seven if and only if the set of these four numbers has at most one number in common with each of the sets A, B, C. We consider three cases.

Case (i). If $a = b = c = d$, then no pair of these numbers sums to 7. This can happen in 6 ways.

Case (ii). If $S = \{a, b, c, d\}$ consists of two distinct values (and no two of these values sum to 7), then these two values are from two of the sets A, B, C. This pair of values can be selected in 12 ways:

$$\{1, 2\}, \quad \{1, 5\}, \quad \{1, 3\}, \quad \{1, 4\}, \quad \{6, 2\}, \quad \{6, 5\}, \quad \{6, 3\}, \quad \{6, 4\}, \quad \{2, 3\},$$

$$\{2,\,4\},\quad \{5,\,3\},\quad \{5,4\}.$$

It can be the case that one number of a pair occurs on three dice and the other number appears on the remaining die. For a given pair there are 2 ways to select the number that occurs three times, and $\binom{4}{1}$ ways to choose which die has the unique value, for a total of

$$2\binom{4}{1} = 8 \text{ ways},$$

It can also happen that each number of a pair occurs on two of the dice. There are then

$$\binom{4}{2} = 6 \text{ ways}$$

to distribute the values on the 4 dice. Thus *Case (ii)* can occur in

$$12(8+6) = 168 \text{ ways}.$$

Case (iii). If $S = \{a,\, b,\, c,\, d\}$ has a value from each of the sets A, B, C, then one of the values must occur on two of the dice and the other two each occur on one die. There are $2^3 = 8$ ways to select one number from each of A, B, C. Once this is done, there are three ways to select the number that will occur on two of the dice. The two dice with the same value can be selected in $\binom{4}{2} = 6$ ways and the remaining two numbers can then be placed in 2 ways. This accounts for

$$8 \cdot 3 \cdot 6 \cdot 2 = 288$$

possibilities,

Therefore the probability that no pair of dice sum to 7 is

$$\frac{4+168+288}{6^4} = \frac{77}{216},$$

so the probability that at least one pair of dice sum to 7 is

$$1 - \frac{77}{216} = \frac{139}{216}.$$

5. **Don't You ♡ Trig Identities?** Let a, b, c be real numbers and suppose that

$$\frac{\sin a + \sin b + \sin c}{\sin(a+b+c)} = \frac{\cos a + \cos b + \cos c}{\cos(a+b+c)}.$$

What is the maximum possible value of $\sin(a+b) + \sin(b+c) + \sin(c+a)$?

Solution 5. Clearing denominators and regrouping we have

$$0 = (\sin(a+b+c)\cos a - \cos(a+b+c)\sin a) + (\sin(a+b+c)\cos b - \cos(a+b+c)\sin b) + (\sin(a+b+c)\cos c - \cos(a+b+c)\sin c) \tag{1}$$

Applying the identity

$$\sin A \cos B - \cos A \sin B = \sin(A - B)$$

to each of the three groupings in (1) we find

$$0 = \sin((a+b+c)-a)+\sin((a+b+c)-b)+\sin((a+b+c)-c) = \sin(b+c)+\sin(a+c)+\sin(a+b).$$

Thus, under the conditions given, $\sin(a+b) + \sin(b+c) + \sin(c+a)$ is always 0.

6. **Population Explosion.** At the end of their first year of farming John and Joanna find that they have 50 cows and 84 pigs. At the end of the second year the tallies are 61 cows and 105 pigs. From that point on, the population of cows at the end of a year is equal to the sum of the cow populations of the previous two years, and the pig population at the end of a year is equal to 10 *more* than the sum of the pig populations of the previous two years. After one of the year-end inventories, John excitedly tells Joanna "If this keeps up, then we will eventually have at least twice as many pigs as cows!" Joanna (who was a math/ag sciences double major) thinks a moment and replies "Ummm, I don't think that's true." Which of John and Joanna is correct? Be sure to justify your answer.

Solution 6. Joanna is correct. Let C_n denote the number of cows and P_n the number of pigs at the end of the n^{th} year of farming. Then

$$C_1 = 50, \quad C_2 = 61, \quad \text{and} \quad C_n = C_{n-2} + C_{n-1}, \quad n \geq 3,$$

and

$$P_1 = 84, \quad P_2 = 105, \quad \text{and} \quad P_n = P_{n-2} + P_{n-1} + 10 \quad n \geq 3.$$

The key to proving that Joanna is correct is to note that

$$P_1 < 2C_1 - 10 \qquad \text{and} \qquad P_2 < 2C_2 - 10.$$

With these initial conditions in place it will always be the case that $P_n < 2C_n - 10$, so the number of pigs will always be less than twice the number of cows. We prove this by induction. Assume that

$$P_n < 2C_n - 10 \qquad \text{and} \qquad P_{n+1} < 2C_{n+1} - 10. \tag{2}$$

Then

$$P_{n+2} = P_n + P_{n+1} + 10 < (2C_n - 10) + (2C_{n+1} - 10) + 10 = 2C_{n+2} - 10.$$

Because (2) is true for $n = 1$, it follows by induction that $P_n < 2C_n - 10$ for all n. Way to go Joanna!

7. **Integral Power!!!!** Let n be a fixed nonnegative integer. For nonzero integer b, define

$$I(b) = I_n(b) = \int_0^{2\pi} \sin^n(bx)\, dx.$$

Prove that the value of $I(b)$ is independent of the nonzero integer b.

Solution 7. It is clear that if $n = 0$, then $I_0(b) = 2\pi$ for all $b \neq 0$.

If n is odd, then make the substitution $x = 2\pi - u$ to obtain

$$I_n(b) = \int_0^{2\pi} \sin^n(bx)\, dx = \int_0^{2\pi} \sin^n(2\pi b - bu)\, du = \int_0^{2\pi} \sin^n(-bu)\, du = -I_n(b).$$

Therefore for odd n, $I_n(b) = 0$ for all integer b.

If n is even, make the substitution $u = bx$ to find

$$I_n(b) = \int_0^{2\pi} \sin^n(bx)\, dx = \frac{1}{b}\int_0^{2\pi b} \sin^n(u)\, du = \begin{cases} \dfrac{b}{b}\displaystyle\int_0^{2\pi} \sin^n(u)\, du & b > 0 \\ -\dfrac{|b|}{b}\displaystyle\int_0^{2\pi} \sin^n(u)\, du & b < 0. \end{cases}$$

The last equality follows because b is an integer and $\sin^n u$ is 2π-periodic. In both cases ($b > 0$ or $b < 0$) the result is equal to

$$\int_0^{2\pi} \sin^n u\, du,$$

so is independent of the nonzero integer b.

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8. **Age Discrimination?** One hundred boys and one hundred girls attend a school dance. For the last dance of the night, every boy dances with exactly one girl and

every girl dances with exactly one boy. As it happened, for each dancing pair, the age difference between the boy and the girl was less than 15 days. Prove that if the boys are lined up in order of youngest to oldest and the girls are lined up in order from youngest to oldest, then the age difference between the m^{th} oldest boy and the m^{th} oldest girl is less than 15 days for $1 \leq m \leq 100$.

Solution 8. We prove the result by contradiction. Let b_k denote the k^{th} boy in the line up and let r_k denote his age, and let g_k denote the k^{th} in the line up and s_k her age. Suppose that there is an integer j, $1 \leq j \leq 100$ for which $|r_j - s_j| \geq 15$ days.

Assume first that $r_j - s_j \geq 15$. Then for the last dance b_j must have been paired with a girl older than g_j, that is, with a girl g_i with $j + 1 \leq i \leq 100$. Because $b_{j+1}, b_{j+2}, \ldots, b_{100}$ are each at least as old as b_i, each of these boys must also have danced with on of the girls $g_{j+1}, \ldots, g_{100}$. Therefore each of the boys in the set $\{b_j, b_{j+1}, \ldots, b_{100}\}$ danced the last dance with one of the girls in the set $\{g_{j+1}, \ldots, g_{100}\}$. However this is impossible because the two sets have a different number of elements.

By a similar argument, $s_j - r_j \geq 15$ is also impossible. Therefore, $|s_i - r_i| < 15$ for $1 \leq i \leq 100$.

9. **Don't be Square!** Let n be a positive integer and let d be a positive divisor of $2n^2$. Prove that $n^2 + d$ is not a perfect square.

Solution 9. We prove the statement by contradiction. Because d is a factor of $2n^2$, there is a positive integer k such that $2n^2 = dk$. If $n^2 + d = m^2$ for some integer m, then

$$m^2 = n^2 + d = n^2 + \frac{2n^2}{k}.$$

We then have

$$m^2k^2 = n^2k^2 + 2n^2k = n^2(k^2 + 2k),$$

so $k^2 + 2k$ must also be a perfect square. However, because k is positive,

$$k^2 < k^2 + 2k < (k+1)^2,$$

so $k^2 + 2k$ cannot be a perfect square. Therefore, $n^2 + d$ cannot be a perfect square.

10. **Its a Sum-thing.** Let $x_1, x_2, \ldots, x_{2021}$ be positive integers with

$$x_1 + x_2 + \cdots + x_{2021} = 3000.$$

Find the maximum and minimum values of

$$x_1^2 + x_2^2 + \cdots + x_{2021}^2.$$

Be sure to justify your answers.

Solution 10. The maximum value for the sum of the squares is $2020 + 980^2$ and the minimum value is $1042 + 4 \cdot 979$.

The maximum value for the sum of the squares is attained when all but one of the $x_i s$ is equal to 1 and the remaining one is equal to 980. To prove this note that if $a,\, b \geq 2$ are positive integers, then

$$a^2 + b^2 < a^2 + b^2 + 2(a-1)(b-1) = 1 + (a+b-1)^2.$$

Thus if $x_j,\ x_k > 1$, then replacing x_j with 1 and x_k with $x_j + x_k - 1$ does not affect $\sum_{i=1}^{2021} x_i$ but does increase the value of $\sum_{i=1}^{2021} x_i^2$. This shows that if two or more of the x_k's are *not* 1, then the sum of the squares is not maximal. Because there are finitely many possible sums of the squares, a maximum value must exist and the sum of the squares is maximized when 2020 of the x_i's equal 1 and the remaining $x_i = 3000 - 2020 = 980$. In this case

$$x_1^2 + x_2^2 + \cdots + x_{2021}^2 = 2020 + 980^2.$$

The minimum value for the sum is attained when 979 of the x_i's are 2 and all of the others are 1. To prove this first note that if $a,\, b$ are positive integers and $a - b > 1$, then

$$a^2 + b^2 > a^2 + b^2 - 2(a-b-1) = (a-1)^2 + (b+1)^2.$$

Thus if $x_k - x_j > 1$, then replacing x_j by $x_j + 1$ and x_k by $x_k - 1$ does not affect $\sum_{i=1}^{2021} x_i$, but does decrease the value of $\sum_{i=1}^{2021} x_i^2$. Therefore, if $x_k - x_j > 1$ for some $j,\, k$, then the sum of the squares is *not* minimal. Because at least one of the x_i's must be 1, it follows that minimum of the the sum of square occurs when each x_i is 1 or 2. Thus the sum of the squares is minimized when 979 of the x_i are equal to 2 and the remaining 1042 are equal to 1. In this case

$$x_1^2 + x_2^2 + \cdots + x_{2021}^2 = 1042 + 4 \cdot 979.$$

Extra Challenge: Under the conditions of the problem, how many different values can the sum of squares take on? What is the second largest value for this sum? The second smallest?