27 th Annual Iowa Collegiate Mathematics Competition

Socially Distanced Locations   March 20, 2021
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For complete credit, all work and reasoning must be clearly communicated.

1. **Marbelous!** A bag contains some marbles, each of which is either red or blue. If one red marble is removed, then $\frac{1}{7}$ of the remaining marbles are red. If a red marble is not removed but instead two blue marbles are removed, then $\frac{1}{5}$ of the remaining marbles are red. How many marbles were in the bag originally? Be sure to justify your answer.

2. **Goldbach Lite.** Prove that every positive integer greater than 6 can be written as a sum of two relatively prime integers, both of which are greater than 1. (Two positive integers are relatively prime if the only positive integer that is a factor of both numbers is 1.)

3. **Stonework.** A two player game starts with a pile of 100 stones. For a turn a player selects one of the piles of stones and divides in into two nonempty piles, but with the provision that a pile with just two stones or one stone cannot be divided. The winner is the last person to make a legal move. Which player has a winning strategy and what is one such strategy?

4. **Are You Feeling Lucky?.** The game *Cant Stop*© is played with four standard six-sided dice. On a turn, a player rolls the four dice, then sums pairs of the dice. What is the probability that *at least* one pair of the four dice sums to seven?

5. **Don't You ♡ Trig Identities?** Let a, b, c be real numbers and suppose that

$$\frac{\sin a + \sin b + \sin c}{\sin(a + b + c)} = \frac{\cos a + \cos b + \cos c}{\cos(a + b + c)}.$$

What is the maximum possible value of $\sin(a + b) + \sin(b + c) + \sin(c + a)$?

6. **Population Explosion.** At the end of their first year of farming John and Joanna find that they have 50 cows and 84 pigs. At the end of the second year the tallies are 61 cows and 105 pigs. From that point on, the population of cows at the end of a year is equal to the sum of the cow populations of the previous two years, and the pig population at the end of a year is equal to 10 *more* than the sum of the pig populations of the previous two years. After one of the year-end inventories, John excitedly tells Joanna "If this keeps up, then we will eventually have at least twice as many pigs as cows!" Joanna (who was a math/ag sciences double major) thinks a moment and replies "Ummm, I don't think that's true." Which of John and Joanna is correct? Be sure to justify your answer.

7. **Integral Power!!!!** Let n be a fixed nonnegative integer. For nonzero integer b, define
$$I(b) = I_n(b) = \int_0^{2\pi} \sin^n(bx)\, dx.$$
Prove that the value of $I(b)$ is independent of the nonzero integer b.

8. **Age Discrimination?** One hundred boys and one hundred girls attend a school dance. For the last dance of the night, every boy dances with exactly one girl and every girl dances with exactly one boy. As it happened, for each dancing pair, the age difference between the boy and the girl was less than 15 days. Prove that if the boys are lined up in order of youngest to oldest and the girls are lined up in order from youngest to oldest, then the age difference between the mth oldest boy and the mth oldest girl is less than 15 days for $1 \leq m \leq 100$.

9. **Don't be Square!** Let n be a positive integer and let d be a positive divisor of $2n^2$. Prove that $n^2 + d$ is not a perfect square.

10. **Its a Sum-thing.** Let $x_1, x_2, \ldots, x_{2021}$ be positive integers with
$$x_1 + x_2 + \cdots + x_{2021} = 3000.$$
Find the maximum and minimum values of
$$x_1^2 + x_2^2 + \cdots + x_{2021}^2.$$
Be sure to justify your answers.