26th Iowa Collegiate Mathematics Competition

University of Iowa, Iowa City   Saturday, March 7, 2020.

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1. Improving Fuel Economy.   Alice drives a truck that gets 10 miles per gallon. Bob drives a small car getting 40 mpg. In one year, they drive the same distance. Alice trades in her truck for a 14 mpg model. Bob wants to trade his car in for one getting X mpg, such that his fuel savings over one year will match Alice's. What should X be?

2. Tennis Anyone? Alice, Bob, and Charlie play tennis in sets. Two of them (chosen randomly) play a set and the winner stays on the court for the next set, with the loser replaced by the idle player. At the end of the day, Alice played 15 sets, Bob played 14 sets, and Charlie played 9 sets. Who played in the 13th set?

3. Apologies to Ramanujan. The number 1729 has the property that it is divisible by the sum of its digits. In fact, 1728 has this property as well. Find three consecutive numbers larger than 1729 such that each is divisible by the sum of its decimal digits.

4. Which Integers are Dots? Let $P(k)$ be the assertion: Every integer is a dot product of two arithmetic progressions of integers, each of length k. For example, $P(3)$ is true if and only if every integer n has the form $a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\{a_i\}$ and $\{b_i\}$ are arithmetic progressions. Determine the truth or falsity of $P(3)$ and $P(4)$.

Note. An *arithmetic progression* is any sequence of the form a, $a+b$, $a+2b$, $a+3b$,..., where a and b are real numbers. The *dot product* of two vectors is $(x, y, z, ...) \cdot (X, Y, Z, ...) = xX + yY + zZ + ...$.

5. Climb the Pyramid You're standing next to a pyramid whose base is a square, 80 feet on a side. The distance from each corner of the base to the top of the pyramid is 70 feet. You want to start at some point along the base of the pyramid and walk up one of the faces in a straight line to the top such that your path makes a 45° angle with the ground. How far from a corner of the base should you start? In the figure, the path you seek is shown as a dashed line.
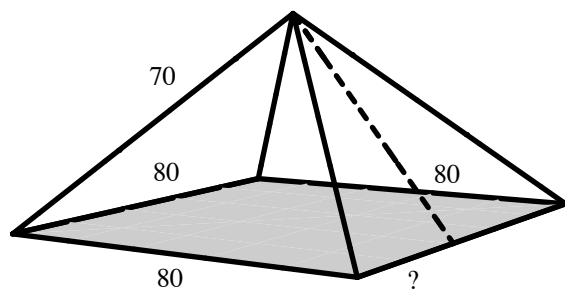
6. A Car and a Goat. Behind three doors labeled 1, 2, 3 are randomly placed a car, a key, and a goat. Alice and Bob win the car if Alice finds the car and Bob finds the key. First Alice (with Bob not present) opens a door; if the car is not behind it she can open a second door. If she fails to find the car, they lose. If she does find the car, then all doors are closed and Bob gets to open one of the three doors in the hope of finding the key and, if not, trying again with a second door. Alice and Bob do not communicate except to make a plan beforehand. What is their best strategy?

7. The Icing on the Cake. A round cake has icing on the top but not the bottom. Cut out a piece in the usual shape (a sector of a circle with vertex at the center), remove it, turn it upside down, and replace it in the cake to restore roundness. Do the same with the next piece; i.e., take a piece with the same vertex angle, but rotated counterclockwise from the first one so that one boundary edge coincides with a boundary edge of the first piece. Remove it, turn it upside-down, and replace it. The cake is self-healing: the cuts disappear once a piece is replaced. Keep doing this in a counterclockwise direction.
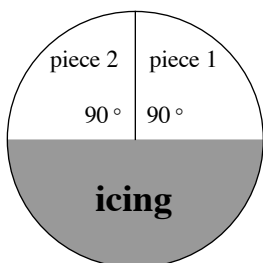
The figure shows the situation after two steps when the central angle is 90°.

If θ is the central angle of the pieces, let $f(\theta)$ be the smallest number of steps needed so that, under the repeated cutting-and-flipping just described, all icing returns to the top of the cake, with $f(\theta)$ set to ∞ if this never happens. For example, $f(90°) = 8$. What is $f(190°)$?

8. A Crosscut Quadrilateral. Bisect the four sides of a convex quadrilateral and connect each midpoint to an opposite corner, choosing the first one in counterclockwise order as in the figure. Prove that the central quadrilateral has the same area as the union of the four corner triangles.
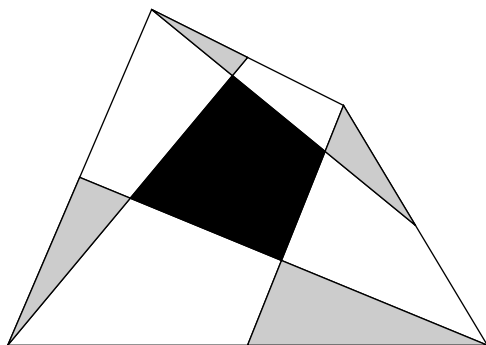
9. Special Numbers. A set A of distinct positive integers is *special above n* if every $x \in A$ such that $x > n$ satisfies:

(1) x divides the product of all $y \in A$ with $y < x$; and (2) x does not divide any $y \in A$ with $y > x$.

For example, {1, 2, 3, 6, 9} is special above 3 and {1, 2, 3, 4, 12} is special above 4 but not special above 3. Find a set of size 9 that is special above 4.

10. Where's Bob? Federal agent Alice is searching for computer hacker Bob, who is hiding in one of 17 caves. The caves form a linear array and every night Bob moves from the cave he is in to one of the caves on either side of it. Alice can search two caves each day, with no restrictions on her choice. For example, if Alice searches (1 2), (2 3), ..., (16 17), then she is certain to catch Bob, though it might take 16 days.

Find a strategy by which Alice is guaranteed to find Bob in 10 days. Partial credit will be given for strategies that require 11, 12, 13, 14, or 15 days.