25th Annual Iowa Collegiate Mathematics Competition

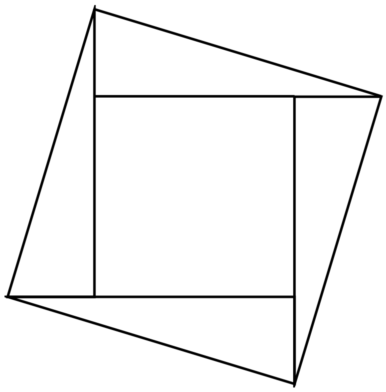
Drake University, Saturday, February 23, 2019

Problems by Razvan Gelca

To receive full credit, all problems require complete justification. Show all your work.

1. Prove that there is no integer that when divided by 6 gives 2 as remainder, and when divided by 15 gives 12 as remainder.

2. Four equal right triangles are arranged as in the figure below to form two squares, one inside the other. It is known that the exterior square has side equal to 41, and the interior square has side equal to 31. Find the sides of the right triangles.



3. How many complex zeros does the polynomial

$$P(z) = z^8 + 2z^6 + 2z^4 - 7z^2 - 8$$

have inside the disk $\{z \in \mathbb{C} \mid |z| < \sqrt[4]{2}\}$?

4. From the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ we randomly choose six numbers. What is the probability that we can write these numbers on the six faces of a cube in such a way that no two adjacent faces contain consecutive numbers? (Two faces are adjacent if they share an edge, one number is written on each face, and the numbers 1 and 9 are considered consecutive.)

5. Let A have coordinates $(2, 3)$ in the plane. Find the coordinates of the points B and C such that the medians from B and C of the triangle ABC have equations $2x + 3y - 1 = 0$ and $2y + 3x - 1 = 0$, respectively.

6. Let

$$f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, \quad f(x) = (x^4 + x^{-4})^4 + (x^4 + x^{-4})^{-4}.$$

Find the intervals on which f is increasing and the intervals on which f is decreasing.

7. Find, with proof, all positive integers x and y such that

$$\sqrt{x} + \sqrt{y} = \sqrt{2772}.$$

8. Compute

$$\int_0^2 \sqrt{x^3+1}\,dx + \int_1^3 \sqrt[3]{x^2-1}\,dx.$$

9. Find the 2020th decimal of the number $(\sqrt{101}+10)^{2019}$.

10. How many triples of integers (a, b, c) are there such that $-2019 < a < b < c < 2019$ and with the property that the lines

$$\begin{aligned} ax + by + c &= 0 \\ bx + cy + a &= 0 \\ cx + ay + b &= 0 \end{aligned}$$

have a common intersection point?