SOLUTIONS for

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The problems are listed in no particular order of difficulty. Each solution requires a proof or justification. Answers only are not enough. Calculators are allowed but certainly not required.

1. Solution: 1/1010 + 1/1011 + 1/1012 + 1/1013 + … + 1/2018

= 1 + 1/2 + 1/3 + … + 1/2018 – (1 + 1/2 + 1/3 + … + 1/1009)

= (1 + 1/3 + 1/5 + … + 1/2017) + (1/2 + 1/4 + 1/6 + … + 1/2018) −2(1/2 + 1/4 + 1/6 + … + 1/2018)

= (1 + 1/3 + 1/5 + … + 1/2017) − (1/2 + 1/4 + 1/6 + … + 1/2018).

2. Solution:

a) Consider the 2^n terms in the expansion of (1+1)(1+2)(1+3) … (1+n) = (n+1)!
They include 1 together with the sum of the $2^n - 1$ members of P(n). Therefore T(P(n)) = (n+1)! – 1.

b) Similarly, consider the 2^n terms in the expansion of
(1+1/1)(1+1/2)(1+1/3) … (1+1/n) = (2/1)(3/2)(4/3) … ((n+1)/n) = n+1 .
They include 1 together with the sum of the $2^n - 1$ members of T(R(n)). Hence T(R(n)) = (n+1) – 1 = n.

Note: Both of these results can also be obtained by using induction.

3. Solution: For x+y = 1, we have f(f(x) + 1 – x) = 2f(1), a constant. Therefore f(x) + 1 – x is also a const., C, so all solutions must be of the form f(x) = x + C – 1 = x + K, where K is a real constant. However, a simple check shows that such functions do not satisfy the original recurrence relation; hence: **no solutions**.

4. Solution: The answer is **no**. Color the sectors alternately black and white. Note that when a checker is moved, the color of its sector changes. When the checkers are moved in pairs, either two checkers in sectors of the same color are moved to two sectors of the other color, or two checkers in sectors of different colors are moved to two sectors of different colors. In either case, the parity of the number of checkers in sectors of either color doesn't change; it either decreases by 2, increases by 2, or stays the same. (BB < -- > WW, BW < -- > WB). Since, initially, there are 3 sectors of each color occupied, there will always be an odd number of checkers in sectors of each color.

5. Solution: Since $P(x) - 1 = (Ax + B)(x-1)^2$ and $P(x) + 1 = (Cx + D)(x+1)^2$, we have

$P'(x) = A(x-1)^2 + 2(Ax + B)(x-1) = C(x+1)^2 + 2(Cx + D)(x+1)$.

Therefore, both $(x-1)$ and $(x+1)$ are factors of the quadratic $P'(x)$. It follows that

$P'(x) = K(x^2-1)$, and so $P(x) = K\left(x^3/3 - x\right) + L$

For $x = 1$, and $x = -1$, we see that $P(1) = 1$ and $P(-1) = -1$.

This leads to $-\frac{2}{3}K + L = 1 - \frac{2}{3}K + L = 1$ and $\frac{2}{3}K + L = -1$,

from which we obtain $K = -3/2$ and $L = 0$. Finally, $P(x) = -x^3/2 + 3x/2$.

6. Solution: Let O be the origin, let P have coordinates (s,s), let S be the point with coordinates $(s,0)$.
Considering the similar right triangles BOA and PSA, we have $\frac{b}{a} = \frac{s}{a-s}$, and therefore $s = \frac{ab}{a+b}$

Similarly(!), considering the similar right triangles B'OA' and PSA', we obtain $s = \frac{a'b'}{a'+b'}$

Note: s is half of the harmonic mean of a and b (and of a' and b').

7. Solution:

a) For $u = x^2 + 1$, we have $du = 2xdx$, $x^2 = u - 1$, and the integral becomes

$$\frac{1}{2}\int \frac{u-1}{u^3}du = \frac{1}{2}\left(-\frac{1}{u} + \frac{1}{2u^2}\right) + C_1 = \frac{-1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C_1 = \frac{-2x^2-1}{4(x^2+1)^2} + C_1$$

b) For $x = \tan\theta$, $dx = sec^2\theta d\theta$, and $x^2 + 1 = sec^2\theta$.

Also, from $x^2 = tan^2\theta = \frac{sin^2\theta}{1-sin^2\theta}$, so $sin^2\theta = \frac{x^2}{1+x^2}$, and the integral becomes

$$\int \frac{tan^3\theta sec^2\theta}{sec^6\theta}d\theta = \int \frac{tan^3\theta}{sec^4\theta}d\theta = \int sin^3\theta\cos\theta d\theta = \frac{sin^4\theta}{4} + C_2 = \frac{x^4}{4(1+x^2)^2} + C_2.$$

c) To reconcile, note that $\frac{x^4}{4(1+x^2)^2} - \frac{-2x^2-1}{4(x^2+1)^2} = \frac{1}{4}$.

8. Solution: Let G_1 be a "maximal" girl -- that is, no girl danced with more boys than G_1. (There could be several "maximal" girls.) Let B_2 be a boy who did not dance with G_1 and let G_2 be a girl who danced with B_2. Among all the boys who danced with G_1 there must be at least one who did not dance with G_2 -- otherwise, G_2 danced with more boys than G_1, and G_1 would not be "maximal". Let B_1 be a boy who danced with G_1 but did not dance with G_2. QED

9. Solution: First, note that there are $C_n^{n+k-1} = C_{k-1}^{n+k-1}$ terms in the expansion of $(a_1 + a_2 + a_3 + \ ... + a_k)^n$

In particular, the expansion of $(a+b+c+d+e)^{10}$ has $C_{10}^{14} = C_4^{14} = 1001$ terms.

a) If the terms containing c's, or d's or e's are eliminated from the expansion of $(a+b+c+d+e)^{10}$, what is left is just the expansion of $(a+b)^{10}$. Therefore number of terms containing neither c's nor d's nor e's is the number of terms in the expansion of $(a+b)^{10}$, which is $C_1^{11} = 11$.

b) Terms in the expansion of $(a+b+c+d+e)^{10}$ containing an a and/or a b are precisely the terms there that are *not* in the expansion of $(c+d+e)^{10}$, and there are $C_2^{12} = 66$ such terms. So the answer is 1001 – 66 = 935.

10. Solution: a) $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$ and $\sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}} > \frac{1}{2\sqrt{n}}$.

b) From a) we have $\sum_{i=2}^{1010^2}\left(\sqrt{i+1} - \sqrt{i}\right) < \sum_{i=2}^{1010^2}\frac{1}{2\sqrt{i}} < \sum_{i=2}^{1010^2}\left(\sqrt{i}\right) - \sqrt{i-1}$

and, after multiplying by 2 and adding 1 throughout, we have

$$1 + 2\sqrt{1010^2+1} - 2\sqrt{2} < \sum_{i=1}^{1010^2}\frac{1}{\sqrt{i}} < 2019$$

But since $1 + 2\sqrt{1010^2+1} - 2\sqrt{2} > 1 + 2020 - 3 = 2018$,

we see that $2018 < \sum_{i=1}^{1010^2}\frac{1}{\sqrt{i}} < 2019$ and the result follows.