Twenty-fourth Annual Iowa Collegiate Mathematics Competition
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The problems are listed in no particular order of difficulty. Each solution requires a proof or justification. Answers only are not enough. Calculators are allowed but certainly not required.

1, Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2018} \quad = \quad \frac{1}{1010} + \frac{1}{1011} + \frac{1}{1012} + \frac{1}{1013} + \ldots + \frac{1}{2018}$$

Hint: This is a special case of a more general result. For all positive integers n,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n} \quad = \quad \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$$

2. For the positive integer n, let S(n) be the set of non-empty subsets of {1, 2, 3,…, n}.

Let P(n) be the collection of the products of the elements of the sets in S(n).

Let R(n) be the collection of reciprocals of the members in P(n).

For a collection of numbers A, let T(A) be the sum of the members of A.

For example, for n=2 we have:

S(2) = {{1}, {2}, {1,2}}; P(2) = <1, 2, 2>; R(2) = <1, ½, ½>; T(P(2)) = 5; and T(R(2)) = 2.

Prove:

a) $T\big(P(n)\big) = (n+1)! - 1$

b) $T\big(R(n)\big) = n$

3. Determine all one-to-one functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + y) = f(x + y) + f(1)$

4. A disc is divided into 6 sectors by three intersecting diameters. One checker is placed in each sector. On a given move, two checkers are moved to neighboring sectors. Can all the checkers end up in one sector?

5. Determine the cubic polynomial P(x) for which $(x-1)^2$ is a factor of P(x) – 1, and $(x+1)^2$ is a factor of P(x) +1.

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6. Let P be a point in the first quadrant on the line $y = x$. Let line L through P intersect the x-axis at point A with coordinates $(a,0)$ and the y-axis at point B with coordinates $(0,b)$. Let line L' through P intersect the x-axis at point A' with coordinates $(a',0)$ and the y-axis at point B' with coordinates $(0,b')$.

Prove that $$\frac{ab}{a+b} = \frac{a'b'}{a'+b'}$$

7. Evaluate the integral $\int \frac{x^3}{(x^2+1)^3}\, dx$ in two ways:

a) by the substitution, $u = x^2 + 1$

b) by the substitution $x = \tan\theta$

c) Explain why the results above are the same.

8. Several boys and girls were at a dance. During the evening, every boy danced with at least one girl, but no girl danced with every boy. Prove that there were two boys, B_1 and B_2, and two girls G_1 and G_2, for which B_1 danced with G_1 and B_2 danced with G_2, but B_1 did not dance with G_2 and B_2 did not dance with G_1.

9. Consider the expansion of $(a+b+c+d+e)^{10}$

a) How many terms contain neither c's, nor d's, nor e's?

b) How many terms contain either an a or a b (or both)?

10. a) Prove that, for positive integers n, $\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}$

b) Prove that $$\left\lfloor \sum_{i=1}^{1010^2} \frac{1}{\sqrt{i}} \right\rfloor = 2018$$

Notation: $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.