Twentieth Annual Iowa Collegiate Mathematics Competition
University of Northern Iowa, Saturday, March 1, 2014

Solutions

1. The closest point on the parabola is the one at which the slope is -2, that is whose $x-$coordinate satisfies $2x - 4 = -2$, so $x = 1$ and the point in question is $(1, 0)$.

2. No, such sequence does not exist. Since a_i are all positive, then $\frac{a_1+a_2+\ldots+a_n}{n} > \frac{a_1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{a_1}{n}$ diverges, so the series in question is also divergent by the Comparison Test.

3. Since 8 and 13 are relatively prime it follows that 8 divides b and 13 divides a and thus $b = 8c$ and $a = 13d$ for some integers c and d. The given condition takes form $8 \cdot 13d = 13 \cdot 8c$ and so $c = d$. Therefore $a + b = 8c + 13c = 21c$, so it is a composite number.

4. Letting $z = a + bi$ and squaring both sides of the given equation we obtain: $a^2+b^2 = 2((a-\sqrt{2})^2+b^2)$, which simplifies to $(a-2\sqrt{2})^2+b^2 = 4$. It follows that the given set A coincides with the circle with the center at $(2\sqrt{2}, 0)$ and radius 2, so the largest value of $|z|$ is $2\sqrt{2} + 2$.

5. Suppose that all the integers are distinct. Then $\frac{1}{\sqrt{n_1}} + \frac{1}{\sqrt{n_2}} + \cdots + \frac{1}{\sqrt{n_{100}}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{100}} < 1 + \int_1^{100} \frac{dx}{\sqrt{x}} = 19$, a contradiction.

6. Let $f(x) = \frac{1}{1+x^3+\sqrt{1+x^6}}$. Write $a(x) = \frac{f(x)+f(-x)}{2}$ and $b(x) = \frac{f(x)-f(-x)}{2}$, so $a(x)$ is even and $b(x)$ is odd and $f(x) = a(x) + b(x)$. Obviously $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} a(x)dx$ as $\int_{-1}^{1} b(x)dx = 0$. Since $a(x) = \frac{1}{2}$, therefore the integral in question is equal to $\int_{-1}^{1} \frac{1}{2}dx = 1$.

7. Label the triangle in a standard way, so the sides a, b and c are opposite to the vertices A, B and C respectively. Let h_a, h_b and h_c be the heights perpendicular to the sides a, b and c respectively, and let S be the area of the triangle and r the radius of the incircle. Then we have: $2S = ah_a = bh_b = ch_c = r(a + b + c)$. By the assumption there exist positive integers x, y and z such that $h_a = xr, h_b = yr$ and $h_c = zr$. Since

$ah_a = r(a+b+c) > 2ra$ it follows that $h_a > 2r$, so $x \geq 3$, and similarly $y \geq 3$ and $z \geq 3$. We also obtain $r = \frac{2S}{a+b+c} = \frac{2S}{\frac{2S}{xr}+\frac{2S}{yr}+\frac{2S}{zr}} = \frac{r}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$ and therefore $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$. It is possible only if $x = y = z = 3$, so the triangle is equilateral.

8. Observe that 2014 is divisible by 2 and not by 4, so if the integers a, b, c and d exist, then exactly one of $a+b, b+c, c+d$ or $d+a$ is even and the other three must be odd, so the sum of all four factors must be odd. On the other hand $(a+b)+(b+c)+(c+d)+(d+a) = 2(a+b+c+d)$, which is a contradiction, so the integers in question do not exist.

9. Let $x_k, 2 \leq k \leq 6$ denote the number of tokens having the value of k dollars. Observe that if either $x_2 + x_4 \geq 30$ or $x_3 + x_6 \geq 20$ or $x_5 \geq 12$, then we can easily form \$60 value. On the other hand if none of them holds then $x_2 + x_4 \leq 29$ and $x_3 + x_6 \leq 19$ and $x_5 \leq 11$, so $x_2 + x_3 + x_4 + x_5 + x_6 \leq 59$, a contradiction.

10. Define the matrix B as $A = B + I$, where I is 4×4 identity matrix. Thus

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

and

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Also $B^n = [0]$ for all $n \geq 3$. Therefore for $n \geq 2$ we have $A^n = (I+B)^n = I^n + \binom{n}{1} I^{n-1} B + \binom{n}{2} I^{n-2} B^2$.It follows that

$$A^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \binom{n}{1} & 1 & 0 & 0 \\ \binom{n}{2} & \binom{n}{1} & 1 & 0 \\ -\binom{n}{1} - \binom{n}{2} & -\binom{n}{1} & 0 & 1 \end{pmatrix}$$

Thus $F(n) = 2\binom{n}{2} + 4\binom{n}{1} + 4 = n^2 + 3n + 4$ and the smallest integer for which $F(n) \geq 2014$ is $n = 44$.