

NINETEENTH ANNUAL
IOWA COLLEGIATE MATHEMATICS COMPETITION

IOWA STATE UNIVERSITY

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Problems and Solutions

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1 Sibling Rivalry

Suppose there are an infinite number of pool balls, which each have a positive integer written on it. For each integer label, there is an infinite supply of balls with that label.

You have a box which contains finitely many such balls. (For example, it may have six #3 balls and twelve #673 balls and a million #2 balls.) Your goal is to empty the box. You may remove any *one* ball you want at each turn. However, whenever you remove a ball, your little brother gets a turn, and he is allowed to add *any finite amount* of balls to the box, as long as each one has a lower number than the one you removed. For example, if you remove one #3 ball, your brother can replace it with 50 #2 balls and 2013 #1 balls.

However, if you remove a #1 ball, then your brother cannot add anything, since there are no balls with lower numbers.

Is it possible to empty the box in a finite number of turns?

Solution: Yes, you can empty the box in a finite number of turns. Use this strategy: always remove a ball with the *highest* label number. For example, suppose that at the beginning, the highest label is #477, and there are 75 such balls. If you remove a #477, your brother cannot replace it with a label of equal or higher number. It follows that after 75 turns, you will no longer have any #477 balls in the box. You may have many balls to contend with, but successive application of the “remove the highest-labeled ball” strategy will eventually empty the box. ■

2 A Sequence of Squares

Prove that every number in the sequence

$$49, 4489, 444889, 44448889, \dots$$

is a perfect square.

Solution: The n th term in the sequence is equal to

$$4 \cdot 10^n \frac{10^n - 1}{9} + 8 \frac{10^n - 1}{9} + 1 = \frac{4 \cdot 10^{2n} - 4 \cdot 10^n + 8 \cdot 10^n - 8 + 9}{9}$$

$$\begin{aligned}
 &= \frac{4 \cdot 10^{2n} + 4 \cdot 10^n + 1}{9} \\
 &= \frac{(2 \cdot 10^n + 1)^2}{9},
 \end{aligned}$$

clearly a perfect square. ■

3 Spin Room

A fan with four equally spaced blades is spinning at 100 revolutions per second clockwise. The fan is in a room that is only illuminated by a strobe light, which flashes (for a tiny instant) 48 times per second. What is the *apparent* speed of the fan, when viewed by a person in the room? In other words, how many revolutions per second does the person actually see?

Solution: The answer is 4 revolutions per second, clockwise. In $1/48$ seconds, the wheel spins $100/48 = 4\frac{1}{12}$ turns, so we perceive $1/12$ of a complete clockwise turn. Hence in one, second, we perceive $48/12 = 4$ complete clockwise turns. ■

4 Polynomial Parity

Let $f(x)$ be a polynomial with integer coefficients satisfying $f(2013) = 2012$. Assume that $f(x)$ can be factored into five polynomials with integer coefficients:

$$f(x) = g_1(x)g_2(x)g_3(x)g_4(x)g_5(x).$$

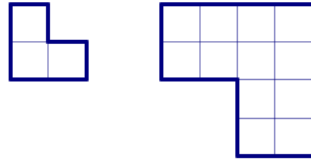
Prove that the sum of the coefficients of at least one factor $g_i(x)$ is odd.

Solution: Assume to the contrary that the sum of the coefficients of every factor is even. The sum of the coefficients of any polynomial $p(x)$ is equal to $p(1)$. Observe that $p(1)$ and $p(2013)$ have the same parity, provided that p has integer coefficients. This can be proven by reducing everything modulo 2, or directly by using the fact that $2013^n - 1^n$ is even, so 2013^n has the same parity as 1^n .

So if the sum of the coefficients of each factor is even, then $g_1(1), g_2(1), \dots, g_5(1)$ are all even, hence $g_1(2013), g_2(2013), \dots, g_5(2013)$ are all even, hence $f(2013) = g_1(2013)g_2(2013) \cdots g_5(2013)$ must be a multiple of 32, contradicting the fact that $2012 = 4 \cdot 503$. ■

5 A Big Ell

Define a *size- n tromino* to be the shape you get when you remove one quadrant from a $2n \times 2n$ square. In the figure below, a size-1 tromino is on the left and a size-2 tromino is on the right.

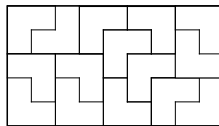


We say that a shape can be *tiled with size-1 trominos* if we can cover the entire area of the shape—and *no excess area*—with *non-overlapping* size-1 trominos. For example, it is easy to see that a 2×3 rectangle can be tiled with size-1 trominos, but a 3×3 square cannot be tiled with size-1 trominos.

Can a size-2013 tromino be tiled by size-1 trominos?

Solution: Yes, you can! There are many possible solutions. One approach uses the following facts. We will abbreviate “tile with size-1 trominos” with “tile.”

1. Since one can tile a 2×3 rectangle, one can also tile a 6×2 rectangle, and a 6×3 , and thus inductively one can tile any $6 \times n$ rectangle for $n \geq 2$.
2. Also, we can tile a 4×9 rectangle.
3. One can tile—somewhat unexpectedly!—a 5×9 rectangle:



4. Using #2 and #3, we can also tile a 9×9 square.

Now we tile the size-2013 tromino by observing that it consists of a size-2007 tromino, plus a border which consists of two 6×2007 rectangles, two 6×4014 rectangles, and three 6×6 squares. All of these border shapes can be tiled using fact #1. For the size-2007 tromino, we observe that 2007 is a multiple of 9, and hence can be tiled by fact #4 (First tile the 2007-tromino with 9×9 squares, each of which can be tiled by size-1 trominos.) ■

6 Playing with a Few Blocks

Is it possible to color the faces of 27 identical $1 \times 1 \times 1$ cubes, using the colors red, white, and blue, so that one can arrange them to form a $3 \times 3 \times 3$ cube with all exterior faces red; and then rearrange them to form a $3 \times 3 \times 3$ cube with all exterior faces blue; and finally, rearrange them to form a $3 \times 3 \times 3$ cube with all exterior faces white?

Solution: Yes, it is possible. We will call the faces of the little cubes “facelets.” Place the $3 \times 3 \times 3$ cube so that one vertex is at $(0,0,0)$ and the opposite vertex at $(3,3,3)$. Now we will color the facelets forming the large cube according to which plane (parallel to the faces of the large cube) that they lie on. Each plane has an equation like $z = 0$ or $x = 2$, etc. In other words, each plane has an equation of the form “variable = constant.” Our rule is simple:

- If the constant is congruent to 0 (mod 3), color all facelets on this plane red.
- If the constant is congruent to 1 (mod 3), color the facelets white.
- If the constant is congruent to 2 (mod 3), color the facelets blue.

For example, along the plane $y = 2$, we color all facelets blue. One can think of this plane as being made out of blue paint, which smears onto each facelet that comes in contact with it.

Why does this work? Imagine filling space with copies of the $3 \times 3 \times 3$ cube. So, for example, there is a new cube lying directly on top of the first cube, with a corner at $(0, 0, 3)$ and an opposite corner at $(3, 3, 6)$, etc. Now continue our color scheme along the infinite planes. For example, all facelets that touch the plane $x = 7$ will be colored white.

If you look at the original cube (the one with a corner at $(0, 0, 0)$), its exterior faces are all colored red. If we translate by the vector $(1, 1, 1)$, we get a new cube whose exterior faces are all colored white. It is easy to see that the parts of this new cube that are outside the first cube can be replaced by the parts of the old cube that were “left behind” under the translation. For example, the new (white cube) contains the small cube with corners $(3, 3, 3)$ and $(4, 4, 4)$. But this small cube is colored exactly the same as the one with corners $(0, 0, 0)$ and $(1, 1, 1)$, etc.

Clearly this “paint by remainder and translate” method will work with n colors on an $n \times n \times n$ cube. ■

7 How Abundant are Abundant Numbers?

An *abundant* number is a positive integer n with $\sigma(n) > 2n$, where $\sigma(n)$ denotes the sum of the divisors of n (including 1 and n). For example, 11 is not abundant, because $\sigma(11) = 1 + 11 = 12 < 2 \cdot 11$, but 18 is abundant, because $\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39 > 2 \cdot 18$.

A positive integer less than a googolplex ($10^{10^{100}}$) is chosen randomly, with all choices equally likely. Let P be the probability that the chosen number is abundant.

- Prove that $P \geq 1/6$ for partial credit;
- Prove that $P > 1/5$ for full credit.

Solution: Using the fact that 6 is a perfect number (the sum of its divisors is exactly two times 6), we deduce that all multiples of 6 (other than 6) are abundant. To see this, let $n = 6k$, where $k > 1$. Then the divisors of $6k$ include $k, 2k, 3k$, and $6k$, which already adds up to $12k$. However, there is at least one other divisor—namely, 1—so the total adds up to more than $12k$.

In a similar way, we can show that all multiples of 20 are abundant (including 20). To see that 20 is abundant, note that its divisors are 1, 2, 4, 5, 10, 20, which add up to $42 > 2 \cdot 20$. If $k > 1$, the divisors of $20k$ will include $k, 2k, 4k, 5k, 10k$, and $20k$, which add up to $42k$.

Thus, at the very least, all numbers that are multiples of 6 or 20 will be abundant (omitting 6 itself, which is “merely” perfect). The fraction of numbers satisfying this is

$$\frac{1}{6} + \frac{1}{20} - \frac{1}{60} = \frac{1}{5}.$$

Since 6 is not abundant, this doesn't give us a probability greater than $1/5$; it is a tiny bit below $1/5$. However, since 28 is a perfect number, we get a bunch of new abundant numbers: all multiples of 28 (other than 28, and also ones that are not also multiples of 60). We don't need to calculate this precisely, but clearly the fraction is greater than $1/5$.

REMARK: The exact "density" of abundant numbers is not known precisely, but it is known to exist, and is slightly below 0.25. ■

8 The Reciprocal of a Semicircle

Let S be the semicircle in the complex plane with center at $1/2$ and radius $1/2$, excluding the left endpoint at the origin. In other notation,

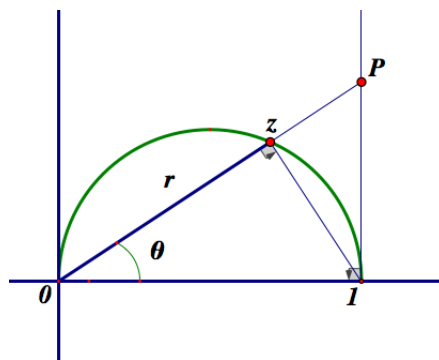
$$S = \left\{ \frac{1}{2} + \frac{1}{2}e^{i\theta} : 0 \leq \theta < \pi \right\}.$$

Describe, with proof, the image of S under the transformation $z \rightarrow 1/z$. In other words, describe the set of points $\{1/z : z \in S\}$.

Solution: The image is the vertical ray $\{1 + it : t \leq 0\}$, perpendicular to the real axis, starting at the complex number $1 = 1 + 0i$. This can be found using rather ugly algebra; we prefer a simple geometrical explanation.

Recall that if $z = re^{i\theta}$, then $1/z = (1/r)e^{-i\theta}$. In other words, the reciprocal of a complex number is another complex number whose magnitude is the reciprocal of the original and whose arg is the negative of the original.

First, let us get a geometric interpretation of reciprocals. In the figure below, z is a point on the given semicircle (whose center is $1/2$ and whose radius is $1/2$) whose arg is θ . Note that the triangle with vertices at $0, z$, and 1 is a right triangle with right angle at z (because it is inscribed in a semicircle).

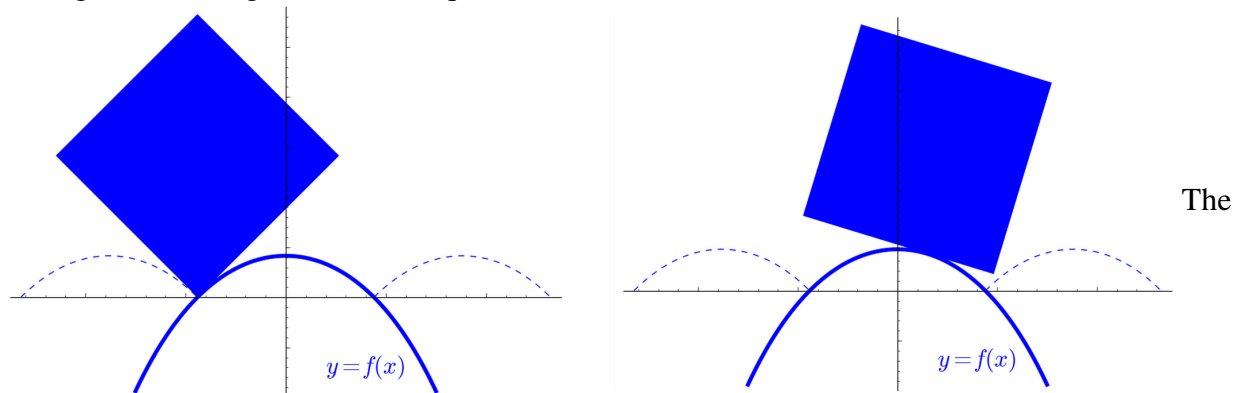


Now draw a line perpendicular to the real axis, going through 1, and let P be the point on this line which is on the ray from 0 to z . We see that triangle $0-z-1$ is similar to triangle $0-1-P$, since this larger triangle is also a right triangle with acute angle θ . Solving proportions, we conclude that the length from 0 to P is equal to $1/r$. Thus P is a complex number whose magnitude is the reciprocal of the magnitude of z , but the arg of P is equal to the arg of z (namely, θ). To get an arg of $-\theta$, we just reflect P with respect to the real axis.

Thus, as the point z moves clockwise on the semicircle from 1 to 0, its image under the map $z \rightarrow 1/z$ will move from $1/1 = 1$ downward on the line $1 + it$, where t ranges from 0 to $-\infty$. ■

9 Square Wheel

Surprisingly, it's possible to get a smooth ride on a square-wheeled bicycle, provided that the riding surface is the correct shape. In the illustration below, a square with side length 2 rotates clockwise around its center, staying tangent to the indicated riding surface, with no slipping, and the center point of the square always lies on the line $y = \sqrt{2}$! The figure on the left shows the starting point, and the figure on the right shows the square after it has traveled some distance.

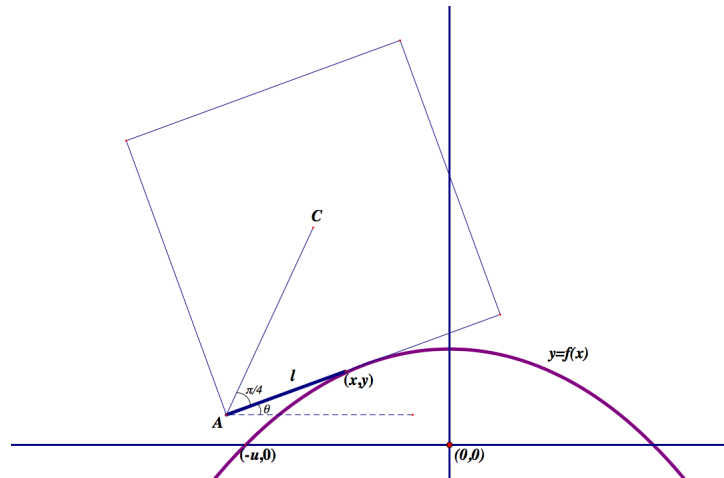


riding surface consists of translated copies of arcs of the part of the graph of $y = f(x)$ that lies above the x -axis. The graph of $y = f(x)$ is the thick-shaded line, and two of the translated arcs are shown with dashed lines.

Find $f(x)$.

Solution: $f(x) = \sqrt{2} - \cosh(x)$, where $\cosh(x)$ is the hyperbolic cosine function defined by $\cosh(x) = (e^x + e^{-x})/2$.

Consider a position of the square wheel, where it is tangent to the curve at (x, y) , with the tangent edge of the wheel making an angle of θ with respect to the x -axis. The wheel began with vertex A at the point $(-u, 0)$. The distance ℓ that wheel traveled along the curve is equal to the arc length from $(-u, 0)$ to (x, y) on this curve, but it is also equal to the length on the square from A to (x, y) .



The other constraint is that the y -coordinate of the center C stays constant and equal to $\sqrt{2}$. Simple vector computations (traveling from (x, y) to A and then to C , keeping track of the y -components) yield

$$\sqrt{2} = y - \ell \sin \theta + \sqrt{2} \sin(\theta + \pi/4).$$

The angle-sum formula simplifies this to

$$\sqrt{2} = y - \ell \sin \theta + \sin \theta + \cos \theta.$$

Differentiate with respect to θ , getting

$$0 = \frac{dy}{d\theta} - \frac{d\ell}{d\theta} \sin \theta - \ell \cos \theta + \cos \theta - \sin \theta. \quad (*)$$

Next, we use the chain rule to get derivatives with respect to x involved, because we can then use the following important facts:

1. $\frac{dy}{dx} = \tan \theta,$

2. $\ell = \int_{-u}^x \sqrt{1 + f'(t)^2} dt$

3. (By differentiating #2 with respect to x):

$$\frac{d\ell}{dx} = \sqrt{1 + f'(x)^2} = \sqrt{1 + \tan^2 \theta} = \sec \theta$$

Applying the chain rule to (*), we have

$$0 = \frac{dy}{dx} \frac{dx}{d\theta} - \frac{d\ell}{dx} \frac{dx}{d\theta} \sin \theta - \ell \cos \theta + \cos \theta - \sin \theta.$$

Using facts #1 and #3, this equation becomes

$$0 = \tan \theta \frac{dx}{d\theta} - \sec \theta \frac{dx}{d\theta} \sin \theta - \ell \cos \theta + \cos \theta - \sin \theta,$$

and since $\sec \theta \sin \theta = \tan \theta$, the first two terms of the above equation cancel out and we are left with

$$-\ell \cos \theta + \cos \theta - \sin \theta = 0.$$

Now divide by $\cos \theta$, getting $1 - \tan \theta = \ell$, and differentiate once more, with respect to θ :

$$-\sec^2 \theta = \frac{d\ell}{d\theta} = \frac{d\ell}{dx} \frac{dx}{d\theta} = \sec \theta \frac{dx}{d\theta},$$

and hence

$$\frac{dx}{d\theta} = -\sec \theta.$$

We can integrate this, getting

$$x = -\ln(\sec \theta + \tan \theta) + C,$$

where C is a constant that is easily seen to equal zero using the condition that when $x = 0$ (at the top of the curve), $\theta = 0$.

We are almost done: Exponentiation yields $e^{-x} = \sec \theta + \tan \theta$, and taking reciprocals (and using the trig identity $\sec^2 \theta - \tan^2 \theta = 1$) yields $e^x = \sec \theta - \tan \theta$. Thus we have

$$\tan \theta = \frac{e^{-x} - e^x}{2}.$$

Since $\tan \theta = dy/dx$, we now integrate with respect to x and get

$$y = -\frac{e^{-x} + e^x}{2} + C,$$

where C is a constant. C can be found by noting that when $x = 0$, $y = \sqrt{2} - 1$ (because the center point of the square has a y -coordinate of $\sqrt{2}$). Thus $C = \sqrt{2}$ and

$$f(x) = y = \sqrt{2} - \frac{e^{-x} + e^x}{2}. \quad \blacksquare$$

10 An Irrational Conclusion

It is well known (and you don't have to prove it) that the set of real numbers forms an infinite-dimensional vector space V over the rationals (i.e., the set of vectors are the real numbers, and the set of scalars are the rational numbers).

Let $2, 3, 5, 7, \dots, p_{2013}$ be the first 2013 prime numbers, and define the vectors $v_k = \ln(p_k)$ for $k = 1, 2, \dots, 2013$. Let S be the subspace of V spanned by the vectors $v_1, v_2, \dots, v_{2013}$. Find, with proof, the dimension of S .

Solution: The dimension is 2013, because these vectors are linearly independent. To see this, suppose, to the contrary, that the vectors were not linearly independent. Then there exist scalars (i.e., rational numbers) $r_1, r_2, \dots, r_{2013}$, not all zero, such that

$$r_1 v_1 + r_2 v_2 + \dots + r_{2013} v_{2013} = 0.$$

Multiplying by the common denominator, we get the equation

$$k_1v_1 + k_2v_2 + \cdots + k_{2013}v_{2013} = 0,$$

where the k_i are all integers, not all zero. Consequently, some of these k_i must be positive, and some must be negative (for if they were all one sign, we couldn't get a sum equal to zero; remember, they cannot all be zero). Ignoring the zero-value k s, and segregating the terms with positive k_i on the left, we get

$$k_{i_1}v_{i_1} + \cdots + k_{i_t}v_{i_t} = \ell_{j_1}v_{j_1} + \cdots + \ell_{j_s}v_{j_s},$$

where the ℓ s are just negatives of the k s that were negative (and hence are positive). Exponentiating both sides of this equation, we get

$$p_{i_1}^{k_{i_1}} \cdots p_{i_t}^{k_{i_t}} = p_{j_1}^{\ell_{j_1}} \cdots p_{j_s}^{\ell_{j_s}}.$$

This equation shows a product using one set of primes $(p_{i_1}, p_{i_2}, \dots, p_{i_t})$ on the left side, which is equal to a product that uses a completely different set of primes $(p_{j_1}, p_{j_2}, \dots, p_{j_s})$ on the right. But this contradicts the Fundamental Theorem of Arithmetic, which states that every integer can be factored *in a unique way* into primes. ■