**Seventeenth Annual Iowa Collegiate Mathematics Competition**

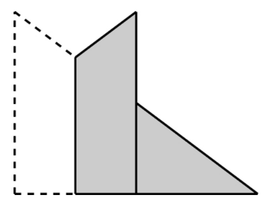
Saturday, February 19, 2011 at Iowa State University

Problems by Sam Vandervelde, svandervelde@stlawu.edu

1. *Finding Factors* Let $p(x) = x^2 + bx + c$ be a quadratic with distinct roots. Compute the degree four polynomial $p(p(x) + x)$ when $p(x) = x^2 + x + 3$, and show that the result is divisible by $x^2 + x + 3$. Then show in general that if $\alpha$ is a root of $p(x)$ then $\alpha$ is also a root of $p(p(x) + x)$.

2. *Questionable Quizzes* Twenty calculus students are comparing grades on their first two quizzes of the year. The class discovers that whenever any pair of students consult with one another, these two students received the same grade on their first quiz or they received the same grade on their second quiz (or both). Prove that the entire class received the same grade on at least one of the two quizzes.

3. *Origami Optimization* Consider a triangular sheet of paper with vertices at the points $(0, 0)$, $(4, 0)$ and $(0, 3)$. By making a single vertical crease in the paper and then folding the left portion of the triangle over the crease so that it overlaps the right portion we obtain a polygon with five sides, colored grey. Find the smallest possible area of this resulting polygon.



4. *Tantalizing Tangents* A smooth curve crosses the $y$-axis at the point $(0, 2)$ and has the following curious property. Given any point $P$ on the curve, the tangent line to the curve at $P$ crosses the $x$-axis at a point $Q$ exactly 2011 units to the right of $P$. (In other words, the $x$-coordinate of $Q$ is 2011 more than the $x$-coordinate of $P$.) Determine the area of the region in the first quadrant bounded by the $x$-axis, the $y$-axis, and this curve, explaining how you found your answer.

5. *Zero to Sixty* A car begins at rest pointing due east. It then steadily accelerates to 60 feet per second (fps), so that its speed at time $t$ seconds is $10t$ fps, for $0 \le t \le 6$. In this time the driver also steadily turns from east to north, so that the car's heading at time $t$ is $15t$ degrees. How far is the car (straight line distance) from its starting point at time $t = 6$ seconds? Give a decimal accurate to four significant digits and indicate how you obtained your answer.

6. *Lively Limits* Compute the exact value of

$$\lim_{t\to\infty} \left( \int_t^{5t} \frac{4x}{x^2 + 2x - 3}\, dx \right).$$

An answer obtained on a calculator is worth partial credit, but it must be justified "by hand" to receive full credit.

7. *Uncertain Seating* Suppose 100 people are each assigned a different seat on an airplane with 100 seats. The passengers are seated one at a time. The first person loses his boarding pass and sits in one of the 100 seats chosen at random. Each subsequent person sits in their assigned seat if it is unoccupied, and otherwise chooses a seat at random from among the remaining empty seats. Determine, with proof, the probability that the last person to board the plane is able to sit in her assigned seat.

8. *Harmonious Rooks* Suppose that the squares of a $2011 \times 2011$ grid are colored alternately black and white in the usual chessboard fashion, so that the four corners are all white. We say that a configuration of rooks on the board is *nonattacking* if no two rooks are in the same row or column. Of all the ways to arrange 2011 nonattacking rooks on the board, what fraction of them involve placing the rooks only on white squares? Write your answer in the form $1/\binom{n}{k}$ for suitable positive integers $n$ and $k$.

9. *Predictable Products* Observe that $(1)(4)(7) = 28$ is one greater than a multiple of 9, while $(2)(5)(8) = 80$ is one less than a multiple of 9. Confirm that this phenomenon persists; in other words, prove that for all $n \geq 1$ we have

$$(1)(4)(7)(10)\cdots(3^n - 2) \equiv 1 \bmod 3^n,$$
$$(2)(5)(8)(11)\cdots(3^n - 1) \equiv -1 \bmod 3^n.$$

10. *Elusive Eigenvalues* Let $A$ be an $n \times n$ matrix of real numbers for $n \geq 3$. Suppose that the upper left entry of $A$ is $n-1$, all other entries in the first column are $-1$, and $A'$ is the $(n-1) \times (n-1)$ submatrix obtained by deleting the left column and top row of $A$. Suppose further that the sum of the entries in each row of $A$ is equal to 0.

a. Show that 0 and $n$ are both eigenvalues of $A$.

b. Prove that if $\lambda \neq 0, 1, n$ is an eigenvalue of $A$ then $\lambda$ is also an eigenvalue of $A'$.