# Sixteenth Annual Iowa Collegiate Mathematics Competition

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## 1 The Mad Veterinarian

In popular culture, many of us are familiar with the stereotype of the mad scientist. In this case, a mad veterinarian invents an animal transmogrifying machine. The machine can transmogrify:

- Two cats into one cat, or vice-versa
- One cat and one dog into one dog, or vice-versa
- Two dogs into one cat, or vice-versa

Beginning with three cats and one dog, is it possible to end up with

(a) one dog and no cats?

(b) one cat and no dogs?

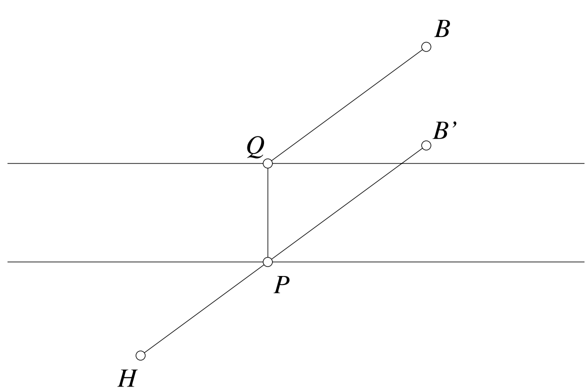
Be sure to justify your answers.

**Solution:** The given transmogrifications always either leave the number of dogs constant or change it by 2. Thus, beginning with an odd number of dogs, the mad veterinarian will always end with an odd number of dogs. It is therefore impossible to end up with one cat and no dogs. To end up with one dog and no cats, the veterinarian can use the second machine three times, among other possibilities.

## 2 The shortest walk

A farmer lives in a farmhouse $H$ on one side of a stream bounded by two parallel lines. He often has to walk to his barn $B$ on the other side of the stream. Since he is tired of getting wet, he wants to build a bridge $PQ$ perpendicular to the stream, with $P$ on the same side of the stream as $H$. He also wants the total walking distance $HP+PQ+QB$ to be as short as possible. How should he determine where to place the bridge?

**Solution:** The farmer should place the bridge such that $HP$ and $QB$ are parallel. To do so, let $w$ be the width of the stream. Translate $B$ toward the stream to a point $B'$ a distance $w$ away from $B$ (with $BB'$ perpendicular to the stream). Then draw $HB'$, and where it intersects the near side of the stream, place point $P$, as shown in the figure.




This distance is minimal since $HP+PQ+QB = HP+PQ+PB'$. $PQ = w$ is constant, and $HP+PB'$ is clearly minimized when P is on the straight line $HB'$.

## 3 Odd divisor sums

What positive integers $n$ have a divisor sum that is odd? For instance, the sum of the divisors of 6 is even since $1+2+3+6=12$, and the sum of the divisors of 9 is odd since $1+3+9=13$.

**Solution:** The sum of the divisors of $n = {p_1}^{a_1} \cdot {p_2}^{a_2} \cdot \ldots$ is

$$\sigma(n) = (1+p_1+p_1^2+\ldots+{p_1}^{a_1}) \cdot (1+p_2+p_2^2+\ldots+{p_2}^{a_2}) \cdot \ldots$$

For this product to be odd, each of the factors must be odd. If $p=2$, then the sum is automatically odd. For any other prime, the sum is odd only when the exponent is even. Thus, the divisor sum is odd for exactly those numbers that are a power of 2 times a perfect square.

## 4 The closest playground

Four families, A, B, C, and D, live in houses at the vertices of a convex quadrilateral.

They decide to put their money together and build a new playground $P$, located at the point where the sum of the distances $PA+PB+PC+PD$ is as small as possible.

Describe how to determine the location of $P$ and prove that the sum of the distances is minimal.

**Solution:** Position $P$ at the intersection of the diagonals of the quadrilateral.

Suppose the diagonals are $AC$ and $BD$. (The same argument works no matter how the four points are arranged.) Then for any point $Q$, $QA+QC \geq AC$, with equality if and only if $Q$ is on the diagonal $AC$. The same inequality holds for the other diagonal. Thus, the minimal distance point must be on both diagonals, and therefore at their intersection.

How does this argument change if the quadrilateral is concave, with for example point $D$ being inside the triangle $ABC$?

## 5 Three-cycles

Find a *real-valued* function $f$ such that the third derivative $f'''(x) = f(x)$ for all $x$, and $f$ is not a constant multiple of $e^x$. Express $f$ in terms of elementary functions (that is, not a power series or limit or other such representation).

**Solution:** Suppose $f(x) = Ae^{kx}$. Then $f'''(x) = k^3Ae^{kx}$. Our desired condition then implies that $k^3 = 1$. Taking $k=1$ would give us a constant multiple of $e^x$, so that's not useful. So let's look at the other two solutions of that cubic, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Unfortunately

$$g(x) = Ae^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} \quad \text{and} \quad h(x) = Ae^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}$$

both give us complex-valued functions.

However, these functions are complex conjugates of each other, so adding them is one way to eliminate the imaginary part. Thus we see that

$$\begin{aligned}
f(x) &= Ae^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} + Ae^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x} \\
&= Ae^{\frac{-x}{2}}\left(e^{i\frac{\sqrt{3}}{2}x} + e^{-i\frac{\sqrt{3}}{2}x}\right) \\
&= Ae^{\frac{-x}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) + i\sin\left(\frac{\sqrt{3}}{2}x\right) + \cos\left(\frac{\sqrt{3}}{2}x\right) - i\sin\left(\frac{\sqrt{3}}{2}x\right)\right) \\
&= 2Ae^{\frac{-x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right).
\end{aligned}$$

Another such function may be found by subtracting these two possibilities and dividing by $i$, rather than adding them, as a way of eliminating the imaginary part from the two complex conjugates. This will result in sin taking the place of cos. Also, of course, the 2 can be absorbed into the $A$ for a slightly simpler solution.

Can you generalize, and find all real-valued functions such that the $n^{\text{th}}$ derivative of $f$ is equal to $f$?

## 6 Four cups, one table

Four empty cups, which can be flipped either up or down, sit at the four compass points on a table. On each round of the game, player 1 chooses which cup(s) to flip, such as "flip the north and west cups". Then player 2 first flips those cups and then rotates the table by any multiple of 90 degrees (including 0) to reposition them. Player 1 wins as soon as all cups are up.

Unfortunately for player 1, the rules require that:

- Player 1 must write down a finite list of instructions for every turn of the game before the game begins.
- Player 2 may look at that list before choosing the initial position of the cups.

Despite this, it is possible for player 1 to guarantee victory! How long is the shortest list of winning instructions for player 1? Give an example of such a list and explain how you can be sure player 1 will win.

**Solution:** An example list is

all, NS, all, NE, all, NS, all, N, all, NS, all, NE, all, NS, all,

and its length is 15.

Clearly no list can be shorter than 15, because such a list cannot visit every state; even with no rotations of the table, there would be an initial state that avoids all the cups being flipped up.

This list suffices because:

- If the four cups are all up, you win before the first move.
- If the four cups are all down, you win after the first move.
- If exactly two opposite cups are up, then after the NS move, either all four cups are up or all four are down, and thus either immediately or after one more flip of all four cups, you win.
- If exactly two adjacent cups are up when the game begins, then that is still true after all, NS, all, and then flipping NE yields a state in which either all four cups have the same state or two opposite cups are up, and thus the repetition of the all, NS, all sequence will guarantee the win.
- If an odd number of cups are up when the game begins, then that is still true after all, NS, all, NE, all, NS, all, and thus the flipping of N yields a state in which an even number of cups are up, and thus a repetition of the initial sequence is guaranteed to win.

In this problem, we had four cups each of which had 2 possible positions. What if we have $n$ cups around a circle, and rather than flipping between 0 and 1 (up and down), they each have $k$ possible positions? For what values of $n$ and $k$ is it possible for player 1 to force a win?

## 7 A minimal area

A smooth function $f(x)$ has $f''(x) > 0$ for all $x$ in $[0,1]$.

For each point $a$ in $[0,1]$, draw the tangent line to $y = f(x)$ at the point where $x = a$. Let $A(a)$ be the area bounded by the curve $y = f(x)$, the tangent line at $a$, $x = 0$, and $x = 1$.

For what value of $a$ is the area minimized?

**Solution:** An equation for the tangent line is $y = f'(a)\cdot(x-a) + f(a)$. Thus we can compute (with $F(x)$ being the antiderivative of $f$ that has $F(0) = 0$):

$$\begin{aligned}
A(a) &= \int_0^1 f(x) - f'(a)(x-a) - f(a)\,dx \\
&= F(1) - f'(a)\left(\frac{(1-a)^2}{2} - \frac{a^2}{2}\right) - f(a) \\
&= F(1) - \frac{f'(a)}{2}\cdot(1-2a) - f(a)
\end{aligned}$$

Thus,

$$A'(a) = 0 - \frac{f''(a)}{2} \cdot (1-2a) + \frac{f'(a)}{2} \cdot 2 - f'(a)$$

Setting $A'(a) = 0$, we see that $a = \frac{1}{2}$ is the solution, no matter what function $f$ is! Checking, we can see that since $f'' > 0$, $A'(a)$ is negative when $a < \frac{1}{2}$ and positive when $a > \frac{1}{2}$, so indeed $a = \frac{1}{2}$ is the absolute minimum on this interval.

## 8 Only somewhat deranged

How many rearrangements of the string of letters *aabcde* have exactly two letters in their original places? The two *a*'s are indistinguishable, so an *a* in either the first or second position is considered to be in its original place.

**Solution:** First choose two positions to be correct.

If the two chosen positions are among the *bcde* spots (that is, positions 3 through 6), then the two *a*'s must be in the remaining two of those positions (and there is only one way to do that), and the remaining two letters must occupy the positions originally held by the *a*'s, and there are two ways to do that. Thus we have $\binom{4}{2} \cdot 1 \cdot 2$ = 12 ways so far.

Otherwise, there are $\binom{6}{2} - \binom{4}{2} = 9$ ways to choose two spots, at least one of which is an *a*. In that case, the remaining 4 spots all contain different letters, and (either by listing the 24 possibilities and counting, or by using the usual derangement arguments) there are $D_4 = 9$ ways to arrange the four letters so that none are in their original spots. Thus there are $9 \cdot 9 = 81$ ways to arrange the letters in this case.

In total we thus have $12 + 81 = 93$ ways.

## 9 Coins in Twoland

In Twoland, the government mints one type of coin for each power of 2, and the base unit of their currency is called the tooie. They also have some unusual laws that require all purchases to be made with exact change, and at most two of each type of coin to be used for any given purchase. For example, to pay 6 tooies, it would be legal to pay with $4+2$ or $4+1+1$ or $2+2+1+1$ but not $2+2+2$ nor 8. It is considered the same way to pay with $4+1+1$ or $1+4+1$ or any other rearrangement of the same coins in a different order; only the collection of coins matters. So there are three ways to pay 6 tooies.

(a) How many ways are there to pay for an item costing 20 tooies?

(b) How many ways are there to pay for an item costing 1000 tooies?

**Solution:** To pay 20 tooies, we can simply list all the possibilities, maybe using a bit of cleverness to help speed things up a bit. If there's a 16, then there's 4 left to pay, and so three ways to do that: 4, $2+2$, and $2+1+1$. If there's no 16, there could be two 8s, and then there's still the same three ways to finish the rest. If there's only one 8, then we need 12 more, which can be made with $4+4+2+2$ or $4+4+2+1+1$. In all there are thus eight ways.

This approach is probably not practical for 1000. One path to the solution is to notice that any odd amount $2n+1$ must be paid with exactly one 1. Then, after removing that 1, all the remaining coins can be divided by 2, thus creating a 1-1 correspondence between ways of paying $n$ and ways of paying $2n+1$. With $f(n)$ denoting the number of ways to pay $n$ tooies, we have $f(2n+1) = f(n)$.

Similarly, for an even amount $2n$, there must be zero or two 1s. If there are zero 1s, divide all the coins by 2 and thus creating a correspondence between ways of paying $n$ and ways of paying $2n$ with zero 1s. If there are two 1s, first remove those and then divide all the coins by 2, thus creating a correspondence between ways of paying $n-1$ and ways of paying $2n$ with two 1s. Thus we have $f(2n) = f(n) + f(n-1)$.

Using those two facts, we have

$$\begin{aligned} f(1000) &= f(500) + f(499) \\ &= (f(250) + f(249)) + f(249) = f(250) + 2f(249) \\ &= (f(125) + f(124)) + 2f(124) = f(125) + 3f(124) \end{aligned}$$

$$\begin{aligned}
&= f(62)+3(f(62)+f(61)) = 4f(62)+3f(61)\\
&= 4(f(31)+f(30))+3f(30) = 4f(31)+7f(30)\\
&= 4f(15)+7(f(15)+f(14)) = 11f(15)+7f(14)\\
&= 11f(7)+7(f(7)+f(6)) = 18f(7)+7f(6)\\
&= 18f(3)+7(f(3)+f(2)) = 25f(3)+7f(2)
\end{aligned}$$

and since $f(3) = 1$ (namely 1+2 as the only way) and $f(2) = 2$ (with 1+1 and 2 as the two ways), we have $25+7\cdot 2 = 39$.

Among other amazing properties of the sequence of integers $f(n)$, Calkin and Wilf pointed out in a recent paper that $\frac{f(n)}{f(n+1)}$ covers every positive rational in lowest terms exactly once.

**10 Square of sum is sum of cubes** Prove that, for any natural number $N$, if $d_1, d_2, \ldots, d_k$ are the divisors of $N$, and $n_1, n_2, \ldots, n_k$ are how many divisors each of $d_1, d_2, \ldots, d_k$ have, then

$$(n_1+n_2+\cdots+n_k)^2 = n_1^3+n_2^3+\cdots+n_k^3.$$

**Solution:** We use strong induction. Suppose it is true for all $N < M$. If $M$ is a prime power, it is the well-known identity for the sum of cubes of consecutive integers. If not, write $M = ab$ where $a$ and $b$ are relatively prime. We use the fact that $d(ab) = d(a)d(b)$, where $d(n)$ is the number of divisors of $n$.

Now observe that for any exponent $t$,

$$\sum_{u|M} d(u)^t = \sum_{e|a,f|b} d(ef)^t = \sum_{e|a,f|b} d(e)^t d(f)^t = \sum_{e|a} d(e)^t \sum_{f|b} d(f)^t \tag{1}$$

and thus

$$\begin{aligned}
\left(\sum_{u|M} d(u)\right)^2 &= \left(\sum_{e|a} d(e) \sum_{f|b} d(f)\right)^2 \quad \text{by (1)}\\
&= \left(\sum_{e|a} d(e)\right)^2 \left(\sum_{f|b} d(f)\right)^2\\
&= \sum_{e|a} d(e)^3 \sum_{f|b} d(f)^3 \quad \text{by inductive hypothesis}\\
&= \sum_{u|M} d(u)^3 \quad \text{by (1)}
\end{aligned}$$

Can you come up with a list of numbers that is not a divisor-counting list as in this problem, but where the square of the sum is still the sum of the cubes? Or are all possible lists of this form?